

# INHOMOGENEOUS CUBIC CONGRUENCES AND RATIONAL POINTS ON DEL PEZZO SURFACES

by

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**Abstract.** — For given non-zero integers  $a, b, q$  we investigate the density of solutions  $(x, y) \in \mathbb{Z}^2$  to the binary cubic congruence  $ax^2 + by^3 \equiv 0 \pmod{q}$ , and use it to establish the Manin conjecture for a singular del Pezzo surface of degree 2 defined over  $\mathbb{Q}$ .

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## 1. Introduction

The quantitative arithmetic of low degree del Pezzo surfaces has received a great deal of attention in recent years. The aim of the present investigation is to provide a new tool in the analysis of such questions and to show how it can be used to estimate the number of  $\mathbb{Q}$ -rational points of bounded height on a del Pezzo surface of degree 2 defined over  $\mathbb{Q}$ . Such surfaces arise as subvarieties  $X \subset \mathbb{P}(2, 1, 1, 1)$  of weighted projective space and are given by equations of the shape

$$x_0^2 + F(x_1, x_2, x_3) = 0,$$

where  $F \in \mathbb{Q}[x_1, x_2, x_3]$  is a quartic form. In the classification of del Pezzo surfaces it is those of degree 1 and 2 whose arithmetic remains the most elusive.

If  $X(\mathbb{Q}) \neq \emptyset$  and  $H : X(\mathbb{Q}) \rightarrow \mathbb{R}_{\geq 0}$  denotes the anticanonical height function then it is natural to study the counting function

$$N_U(B) := \#\{x \in U(\mathbb{Q}) : H(x) \leq B\},$$

as  $B \rightarrow \infty$ , for a Zariski open subset  $U \subseteq X$  obtained by deleting the accumulating subvarieties. These are the exceptional divisors arising from the bitangents of the plane quartic curve

$F(x_1, x_2, x_3) = 0$ . There are 28 of these when  $F$  is non-singular, producing 56 exceptional curves on  $X$ . A well-known conjecture of Manin [15] predicts the existence of constants  $c_X > 0$  and  $\rho_X \in \mathbb{N}$  such that

$$N_U(B) = c_X B(\log B)^{\rho_X - 1} (1 + o(1)), \quad (1.1)$$

as  $B \rightarrow \infty$ . Moreover, if  $\tilde{X}$  denotes the minimal desingularisation of  $X$ , with  $\tilde{X} = X$  if it is non-singular, then it is expected that  $\rho_X = \text{rank}(\text{Pic } \tilde{X})$ , where  $\text{Pic } \tilde{X}$  is the Picard group of  $\tilde{X}$ . There is a prediction of Peyre [21] concerning the value of the constant  $c_X$ . These refined conjectures have received a great deal of attention in the context of del Pezzo surfaces of degree at least 3, an account of which can be found in Browning's treatise [7]. Our success in degree 2 has been rather limited however.

When  $X$  is non-singular it follows from work of Broberg [6] that  $N_U(B) = O_{\varepsilon, X}(B^{9/4+\varepsilon})$  for any  $\varepsilon > 0$ . This argument uses Siegel's lemma to cover the rational points of height at most  $B$  on  $X$  with  $O(B^{3/2})$  plane sections  $H$  defined over  $\mathbb{Q}$ . For each of these one is left with counting points of bounded height on the curves  $C_H \subset \mathbb{P}(2, 1, 1)$  given by

$$y_0^2 + G(y_1, y_2) = 0,$$

with  $G \in \mathbb{Q}[y_1, y_2]$  a binary form of degree 4. This one needs to do uniformly with respect to the coefficients of  $G$  and is achieved via a modification of Heath-Brown's determinant method [16]. For generic  $H$  the curve  $C_H$  has genus 1 and so one expects it to have very few rational points. Nevertheless it is difficult to demonstrate this with the requisite degree of uniformity.

In this paper we will consider split singular  $X$ , by which we mean that the quartic form  $F$  is singular and the singularities and exceptional curves of  $X$  are all defined over  $\mathbb{Q}$ . According to the classification of Alexeev and Nikulin [1] the minimal desingularisation  $\tilde{X}$  can then be realised as the blow-up of  $\mathbb{P}^2$  along 7  $\mathbb{Q}$ -points in "almost general position". As such one verifies that  $\text{Pic } \tilde{X} \cong \mathbb{Z}^8$ , so that  $\rho_X = 8$  in the Manin conjecture. We will impose the condition that  $F$  is reducible and we will assume that  $X \subset \mathbb{P}(2, 1, 1, 1)$  takes the shape

$$x_0^2 + x_1 x_2^3 + x_1^3 x_3 = 0. \quad (1.2)$$

This defines a del Pezzo surface of degree 2 with a unique exceptional curve  $x_0 = x_1 = 0$  and a unique singularity  $[0, 0, 0, 1]$ . According to the classification of Arnol'd [2] a hypersurface has a simple  $\mathbf{E}_7$  singularity if it can be put in the local normal form  $z_1^3 + z_1 z_2^3 + z_3^2 + \dots + z_k^2$ , where  $k - 1$  is the dimension of the hypersurface. Dividing through by  $x_2^4$  and making a change of variables one is directly led to this normal form, so that  $X$  has a simple  $\mathbf{E}_7$  singularity. It follows from work of Ye [26, Lemma 4.6] that there is only one such surface up to isomorphism. We let  $U$  be the Zariski open subset formed by deleting the curve  $x_0 = x_1 = 0$  from  $X$ .

When one returns to the above argument involving plane sections, the situation is more favourable since the resulting curves  $C_H$  generically define elliptic curves with rational 2-torsion. In this way one can show that  $N_U(B) = O_{\varepsilon}(B^{3/2+\varepsilon})$  for any  $\varepsilon > 0$ . Our goal is to show how the full Manin conjecture (1.1) can be established for this particular surface using analytic number theory. We will establish the following result.

**Theorem 1.1.** — *We have*

$$N_U(B) = cB(\log B)^7(1 + o(1)),$$

as  $B \rightarrow \infty$ , where  $c > 0$  is the constant predicted by Peyre.

The investigations of Derenthal and Loughran [10, 12] show that  $X$  is neither toric nor an equivariant compactification of  $\mathbb{G}_a^2$ . Thus Theorem 1.1 is not a special case of [3] or [9]. The proof

of our theorem relies on a passage to the universal torsor above the minimal desingularisation  $\tilde{X}$  of  $X$ , which in this setting is a subset of the affine hypersurface  $T \subset \mathbb{A}^{11}$ , given by the equation

$$\eta_1^2 \eta_2 \alpha_1^3 + \eta_7 \alpha_2^2 + \eta_4 \eta_5^2 \eta_6^3 \eta_8^4 \alpha_3 = 0. \quad (1.3)$$

The idea is to establish a bijection between  $U(\mathbb{Q})$  and a suitable subset of  $T(\mathbb{Z})$ . This step underpins many proofs of the Manin conjecture, such as that found in work of la Bretèche, Browning and Derenthal [5] dealing with the split cubic surface of singularity type  $\mathbf{E}_6$ . Here, as there, it is useful to view the torsor equation as a congruence  $\eta_1^2 \eta_2 \alpha_1^3 + \eta_7 \alpha_2^2 \equiv 0 \pmod{\eta_4 \eta_5^2 \eta_6^3 \eta_8^4}$ , with  $\alpha_1, \alpha_2$  being thought of as the main variables. Regrettably one finds that the arguments developed in [5] no longer bear fruit in the present setting. Rather one is forced to consider in general terms the counting function

$$M(B, \mathbf{X}, \mathbf{Y}; a, b; q) := \# \left\{ (x, y) \in \mathbb{Z}^2 : \begin{array}{l} 0 < x \leq \mathbf{X}, \ |y| \leq \mathbf{Y}, \ (xy, q) = 1, \\ ax^2 + by^3 \equiv 0 \pmod{q}, \ |ax^2 + by^3| \leq qB \end{array} \right\}, \quad (1.4)$$

for  $B \geq 2$ ,  $\mathbf{X}, \mathbf{Y} \geq 1$  and non-zero integers  $a, b, q$  such that  $q > 0$ . In our case of interest we have  $a = \eta_7$ ,  $x = \alpha_2$ ,  $b = \eta_1^2 \eta_2$ ,  $y = \alpha_1$  and  $q = \eta_4 \eta_5^2 \eta_6^3 \eta_8^4$ . One seeks an asymptotic formula for  $M(B, \mathbf{X}, \mathbf{Y}; a, b; q)$  which is completely uniform in the relevant parameters. This will ultimately be achieved in §8.4, where it is recorded as Theorem 8.1. A substantially easier problem is to produce a good upper bound for the counting function in which the condition  $|ax^2 + by^3| \leq qB$  is dropped. Let us denote by  $M(\mathbf{X}, \mathbf{Y}; a, b; q)$  this counting function. During the course of our argument we will be led to the following result in §7, which we record here for ease of use.

**Theorem 1.2.** — *Let  $\varepsilon > 0$ , let  $\mathbf{X}, \mathbf{Y} \geq 1$  and assume that  $(ab, q) = 1$ . Then we have*

$$\begin{aligned} M(\mathbf{X}, \mathbf{Y}; a, b; q) &\ll (q\mathbf{X}\mathbf{Y})^\varepsilon \left\{ \frac{\mathbf{X}\mathbf{Y}}{q} + \frac{a^{1/2}b^{-1/4}\mathbf{X} + a^{1/4}\mathbf{X}^{1/2}\mathbf{Y}^{3/4}}{q^{1/2}} \right. \\ &\quad \left. + (s(q)s_1(q)q)^{1/2} \left( \frac{b^{1/2}\mathbf{Y}}{\mathbf{X}} + \frac{a^{1/2}}{\mathbf{Y}^{1/2}} \right) \right\}, \end{aligned}$$

where

$$s(q) := \prod_{p|q} p, \quad s_1(q) := \prod_{\substack{p^\nu || q \\ 2 \nmid \nu}} p. \quad (1.5)$$

The implied constant in this estimate is allowed to depend on the choice of small parameter  $\varepsilon > 0$ , a convention that we adhere to in all of our estimates. Our estimate for  $M(\mathbf{X}, \mathbf{Y}; a, b; q)$  is sharpest when  $q$  has small square-free kernel or when  $\mathbf{Y}$  is small compared with  $\mathbf{X}$ . It is the cornerstone of our entire investigation. It should be noted that when  $q$  is square-free and  $a = b = 1$  Theorem 1.2 does not give anything sharper than what is available through the work of Pierce [22].

We proceed to give a simplified description of the method behind Theorem 1.2. The  $x, y$  appearing in (1.4) are constrained to lie in a certain region and our first task will be to cover this region by small boxes and to approximate their characteristic functions by smooth weights. This facilitates an application of the Poisson summation formula, which we invoke after breaking the sums over  $x$  and  $y$  into residue classes modulo  $q$ . This transformation leads to expressions involving exponential sums of the form

$$E(m, n; q) = \sum_{\substack{x=1 \\ (x, q)=1}}^q e\left(\frac{mx^3 - nx^2}{q}\right),$$

for  $m, n \in \mathbb{Z}$ , where  $e(\cdot) := \exp(2\pi i \cdot)$ . If one were now to estimate these sums directly one would retrieve an estimate of the sort obtained by Pierce [22]. In the setting of Theorem 1.1, however, this would only yield a final upper bound of the form  $N_U(B) = O_\varepsilon(B^{3/2+\varepsilon})$ . A key point in the method

is to evaluate the sums  $E(m, n; q)$  explicitly for power-full moduli  $q$ , a situation that explicitly arises in the application to Theorem 1.1 since then  $q = \eta_4 \eta_5^2 \eta_6^3 \eta_8^4$  contains high powers. This is carried out in §4, where an easy multiplicativity property renders it sufficient to study  $E(m, n; p^t)$  for primes  $p$  and suitable  $t \geq 2$ . It turns out that considerable labour is required to deal with the primes  $p = 2$  and  $p = 3$  and the reader may be inclined to take the results of this section on faith at a first reading.

After explicitly evaluating the exponential sums  $E(m, n; q)$  at power-full moduli we encounter terms of the shape

$$e\left(\frac{\overline{m}^2 n^3}{q}\right),$$

for  $m, n \in \mathbb{Z}$  such that  $(m, q) = 1$  and where  $\overline{m}$  is the multiplicative inverse of  $m$  modulo  $q$ . We need to sum these up non-trivially. Our second key innovation is to flip the numerator and denominator, using the familiar identity

$$e\left(\frac{\overline{l}}{k}\right) e\left(\frac{\overline{k}}{l}\right) = e\left(\frac{1}{kl}\right), \quad (1.6)$$

for any non-zero integers  $k, l, \overline{k}, \overline{l}$  such that  $k\overline{k} + l\overline{l} = 1$ . This has the desired effect of reducing the size of the denominator drastically. Next, we break up the summation over  $n$  into residue classes modulo the new denominator and use Poisson summation in  $n$  again. This time we encounter new cubic exponential sums and cubic exponential integrals which we need to estimate non-trivially. This leads to an asymptotic estimate for the number of solutions of the congruence  $ax^2 + by^3 \equiv 0 \pmod{q}$  in small boxes. The final step is to sum up all these contributions. In fact this summation is also carried out non-trivially, with parts of the averaging process replaced by an integration, in order to take advantage of extra cancellations. While this leads to substantial extra work it is nonetheless essential for obtaining the asymptotic formula in Theorem 1.1.

Aside from its intrinsic utility in the proof of Theorem 1.1 we can use Theorem 1.2 and its refinement Theorem 8.1 to tackle other questions in Diophantine geometry. Firstly, an inspection of the various universal torsors calculated by Derenthal [10] arising in the theory of split singular del Pezzo surfaces, shows that the underlying counting function precisely matches (1.4) whenever its singularity type is maximal. Although we will not present details here, it is possible to provide independent proofs of the Manin conjecture for such cases, albeit with weaker error terms than are already available. This includes the  $\mathbf{E}_6$  cubic surface considered by la Bretèche, Browning and Derenthal [5], the  $\mathbf{D}_5$  degree 4 del Pezzo surface considered by la Bretèche and Browning [4] and the  $\mathbf{A}_4$  degree 5 del Pezzo surface. The latter two surfaces are equivariant compactifications of  $\mathbb{G}_a^2$  by [12] and so are covered by the investigation of Chambert-Loir and Tschinkel [9].

A second and rather different application of Theorem 1.2 lies in the theory of elliptic curves over  $\mathbb{Q}$ . Such curves may be brought into Weierstrass form

$$E_{A,B} : y^2 = x^3 + Ax + B,$$

for  $A, B \in \mathbb{Z}$  with non-zero discriminant  $-16(4A^3 + 27B^2) = -16\Delta_{A,B}$ , say. It is presently unknown whether there are infinitely many  $E_{A,B}$  for which  $\Delta_{A,B}$  is prime. An easier questions concerns the square-freeness of  $\Delta_{A,B}$ . Let  $\max\{|A|^{1/4}, |B|^{1/6}\}$  be the exponential height of  $E_{A,B}$  and let  $\mu$  be the Möbius function. We maintain the conventions  $\mu(0) = 0$  and  $\mu(-n) = \mu(n)$ . A measure of the density of elliptic curves with square-free discriminant is achieved by studying the quantity

$$S(X) := \sum_{\substack{(A,B) \in \mathbb{Z}^2 \\ |A| \leq X^4, |B| \leq X^6}} \mu^2(\Delta_{A,B}).$$

We will use Theorem 1.2 to establish the following result in §2.

**Theorem 1.3.** — For  $q \in \mathbb{N}$  let  $\varrho(q) := \#\{\alpha, \beta \bmod q : \Delta_{\alpha, \beta} \equiv 0 \bmod q\}$ . Then for any  $\varepsilon > 0$  we have

$$S(X) = 4X^{10} \prod_p \left(1 - \frac{\varrho(p^2)}{p^4}\right) + O(X^{7+\varepsilon}).$$

For comparison Wong [25, Proposition 6] has shown a similar asymptotic formula but with only a logarithmic saving in the error term. In fact, as we shall see in §2, an adaptation of an argument due to Estermann [14] would permit a power saving but only leads to an error term of order  $X^{8+\varepsilon}$ . Through Möbius inversion the problem is to count solutions to the congruence  $\Delta_{A,B} = 4A^3 + 27B^2 \equiv 0 \bmod k^2$  for square-free integers  $k$ . Estermann's approach helps us to deal with the contribution from both small and large  $k$ . Theorem 1.2 is the key ingredient in the treatment of medium  $k$ .

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## 2. Elliptic curves with square-free discriminant

In this section we show how Theorem 1.3 follows from Theorem 1.2. Using the Möbius function to detect the square-freeness condition we may write

$$S(X) = \sum_{k=1}^{\infty} \mu(k) \sum_{\substack{(A,B) \in \mathcal{A}(X) \\ k^2 | 4A^3 + 27B^2}} 1,$$

where  $\mathcal{A}(X) := \{(A, B) \in \mathbb{Z}^2 : |A| \leq X^4, |B| \leq X^6\}$ . We will estimate this inner sum differently according to the size of  $k$ . For parameters  $\xi_1 < \xi_2$  let us write  $S_i(X)$  for the overall contribution to  $S(X)$  from  $k$  in the interval  $I_i$ , with

$$I_1 = (0, \xi_1] \cap \mathbb{N}, \quad I_2 = (\xi_1, \xi_2] \cap \mathbb{N}, \quad I_3 = (\xi_2, \infty) \cap \mathbb{N}.$$

Finally for  $q \in \mathbb{N}$  we recall the definition of  $\varrho(q)$  from Theorem 1.3. This is a multiplicative function of  $q$  and it is easy to see that  $\varrho(p^2) = O(p^2)$  for any prime  $p$ . Hence we have  $\varrho(k^2) = O(k^{2+\varepsilon})$  for any square-free  $k \in \mathbb{N}$ .

Beginning with the contribution from small  $k$ , the idea is to break the sum over  $A, B$  into congruence classes modulo  $k^2$ . Beginning with the sum over  $B$  one sees that

$$S_1(X) = \sum_{k \leq \xi_1} \mu(k) \sum_{|A| \leq X^4} \sum_{\substack{\beta \bmod k^2 \\ \Delta_{A,\beta} \equiv 0 \bmod k^2}} \left( \frac{2X^6}{k^2} + O(1) \right),$$

where we recall that  $\Delta_{A,B} = 4A^3 + 27B^2$ . For  $n \in \mathbb{Z}$  and square-free  $k \in \mathbb{N}$  let  $\gamma_{k^2}(n)$  denote the number of incongruent solutions  $\beta$  modulo  $k^2$  of  $n \equiv 27\beta^2 \bmod k^2$ . It is trivial to see that  $\gamma_{k^2}(n) \ll k^\varepsilon(k, n)$  for any  $\varepsilon > 0$ . We may now write

$$S_1(X) = 2X^6 \sum_{k \leq \xi_1} \frac{\mu(k)}{k^2} \sum_{|A| \leq X^4} \gamma_{k^2}(-4A^3) + O(\xi_1^{2+\varepsilon} + X^4 \xi_1^{1+\varepsilon}),$$

where the error term  $\xi_1^{2+\varepsilon}$  comes from  $A = 0$ . Next we claim that

$$\sum_{|A| \leq z} \sigma_{k^2}(-4A^3) = \frac{2z\rho(k^2)}{k^2} + O(k^{1+\varepsilon}), \quad (2.1)$$

for any square-free  $k \in \mathbb{N}$ . Once armed with this it is then straightforward to see that

$$S_1(X) = 4X^{10} \prod_p \left(1 - \frac{\varrho(p^2)}{p^4}\right) + O\left(\frac{X^{10}}{\xi_1^{1-\varepsilon}} + \xi_1^{2+\varepsilon} + X^4 \xi_1^{1+\varepsilon} + X^6 \xi_1^\varepsilon\right), \quad (2.2)$$

on extending the summation over  $k$  to infinity. We establish the claim using a simple argument involving exponential sums.

Breaking the sum over  $A$  into residue classes modulo  $k^2$  we find that

$$\begin{aligned} \sum_{|A| \leq z} \sigma_{k^2}(-4A^3) &= \sum_{\substack{\alpha, \beta \bmod k^2 \\ \Delta_{\alpha, \beta} \equiv 0 \bmod k^2}} \sum_{\substack{|A| \leq z \\ A \equiv \alpha \bmod k^2}} 1 \\ &= \frac{1}{k^2} \sum_{\ell \bmod k^2} \sum_{\substack{\alpha, \beta \bmod k^2 \\ \Delta_{\alpha, \beta} \equiv 0 \bmod k^2}} \sum_{|A| \leq z} e\left(\frac{\ell(\alpha - A)}{k^2}\right). \end{aligned}$$

The contribution from  $\ell = 0$  is clearly

$$\frac{\rho(k^2)}{k^2} (2z + O(1)) = \frac{2z\rho(k^2)}{k^2} + O(k^\varepsilon),$$

which is satisfactory. Likewise one finds that the contribution from non-zero  $\ell$  is

$$\ll \frac{1}{k^2} \sum_{\substack{-k^2/2 < \ell \leq k^2/2 \\ \ell \neq 0}} \tau(k^2; \ell) \min \left\{ z, \left\| \frac{\ell}{k^2} \right\|^{-1} \right\} \ll \sum_{1 \leq \ell \leq k^2/2} \frac{\tau(k^2; \ell)}{\ell},$$

where

$$\tau(q; \ell) = \sum_{\substack{\alpha, \beta \bmod q \\ \Delta_{\alpha, \beta} \equiv 0 \bmod q}} e\left(\frac{\ell\alpha}{q}\right).$$

The exponential sum  $\tau(q; \ell)$  satisfies a basic multiplicativity property in  $q$  rendering it sufficient to understand when  $q$  is a prime power. In this way one easily concludes that  $\tau(k^2; \ell) \ll k^{1+\varepsilon}(k, \ell)$ . Thus the contribution from non-zero  $\ell$  is also seen to be satisfactory for (2.1), after redefining  $\varepsilon$ .

Turning to the large values of  $k$  we write the congruence as an equation and note that

$$|S_3(X)| \leq \sum_{0 < |m| \leq X^{12}/\xi_2^2} \# \{(A, B, k) \in \mathcal{A}(X) \times I_3 : 4A^3 + 27B^2 = k^2 m\}.$$

To estimate the summand we fix  $A$  and consider it as a problem about counting the representations of  $4A^3$  by the binary quadratic form  $mx^2 - 27y^2$ . The classical argument of Estermann [14] shows that there are  $O((Am)^\varepsilon)$  solutions  $(x, y)$  for any  $\varepsilon > 0$ , whence

$$S_3(X) \ll \frac{X^{16+\varepsilon}}{\xi_2^2}. \quad (2.3)$$

At this point we can recover a preliminary estimate for  $S(X)$  by taking  $\xi_1 = \xi_2 = X^4$ . This gives a version of Theorem 1.3 with the weaker error term  $O(X^{8+\varepsilon})$ , as remarked in the introduction.

We now come to the treatment of the middle range for  $k$ . Thus we have

$$|S_2(X)| \leq \sum_{\xi_1 < k \leq \xi_2} \mu^2(k) \# \{(A, B) \in \mathcal{A}(X) : 4A^3 + 27B^2 \equiv 0 \bmod k^2\}.$$

For given  $k$  let us write  $k = k_2 k_3 k'$  where  $k_2 = (k, 2)$ ,  $k_3 = (k, 3)$  and  $k'$  is coprime to 6. It readily follows that  $k_2 \mid B$  and  $k_3 \mid A$  in the summand. Making the change of variables  $A = k_3 A'$  and  $B = k_2 B'$  we deduce that

$$|S_2(X)| \leq \sum_{\substack{\xi_1 < k \leq \xi_2 \\ k = k_2 k_3 k'}} \mu^2(k) \# \left\{ (x, y) \in \mathbb{Z}^2 : \begin{array}{l} k_3 |A'| \leq X^4, \quad k_2 |B'| \leq X^6 \\ a^3 B'^2 + b^2 A'^3 \equiv 0 \pmod{k'^2} \end{array} \right\},$$

where  $a = 3/k_3$  and  $b = 2/k_2$ . In particular it follows that  $(ab, k') = 1$  and  $a, b \leq 3$ . We will need to account for possible common factors of  $A'B'$  and  $k'^2$ . Drawing out the greatest common divisor of  $B'$  and  $k'$  we write  $B' = hx$  and  $k' = h\ell$ , with  $(x, \ell) = 1$ . It easily follows from the square-freeness of  $k$  that  $h \mid A'$  and we can write  $A' = hy$  with  $(hxy, \ell) = 1$ . Hence

$$|S_2(X)| \leq \sum_{\substack{\xi_1 < k \leq \xi_2 \\ k = k_2 k_3 h \ell}} \mu^2(k) M \left( \frac{X^6}{h}, \frac{X^4}{h}; a^3, b^2 h; \ell^2 \right),$$

in the notation of §1, with  $(a^3 b^2 h, \ell^2) = 1$ .

Everything is now in place for an application of Theorem 1.2. On noting that  $s(\ell^2) = s_1(\ell^2) = 1$ , we deduce that

$$M \left( \frac{X^6}{h}, \frac{X^4}{h}; a^3, b^2 h; \ell^2 \right) \ll (\ell X)^\varepsilon \left\{ \frac{X^{10}}{h^2 \ell^2} + \frac{X^6}{h \ell} + \frac{h^{1/2} \ell}{X^2} \right\}.$$

Inserting this into our bound for  $S_2(X)$  we conclude that

$$S_2(X) \ll (\xi_2 X)^\varepsilon \left\{ \frac{X^{10}}{\xi_1} + X^6 + \frac{\xi_2^2}{X^2} \right\}.$$

We must now combine this with (2.2), (2.3) and a suitable choice of  $\xi_1, \xi_2$ . Taking  $\xi_1 = X^3$  and  $\xi_2 = X^{9/2}$  readily leads to the statement of Theorem 1.3.

### 3. Technical preliminaries

**3.1. Gaussian weights.** — There are numerous ways in which one can approximate the characteristic function of intervals using smooth weights. In our work, which will involve repeated applications of the Poisson summation formula, it will be useful to have weights which transform well under the Fourier transform. We are naturally drawn to construct weight functions from the Gaussian

$$\Gamma(x) := \exp(-\pi x^2). \quad (3.1)$$

Note that the usual Gamma function does not occur anywhere in our work and so we trust that this choice of notation doesn't cause confusion. Let  $\chi$  denote the characteristic function of the interval  $[-1/2, 1/2)$ . We will approximate this using the weights

$$\Phi_\pm(x) := L \int_{-1/2 \mp \Delta^{1/2}}^{1/2 \pm \Delta^{1/2}} \Gamma((x - \mu)L) d\mu \pm \varepsilon_1 \Gamma(x),$$

for any  $L > 2$ , where  $\Delta := L^{-1}$  and  $\varepsilon_1 := 24 \exp(-L)$ . Clearly,  $\Phi_+$  and  $\Phi_-$  are smooth (infinitely differentiable) functions that have rapid decay at  $-\infty$  and  $\infty$ . Thus  $\Phi_\pm(x) = O(x^{-N})$  as  $|x| \rightarrow \infty$  for any fixed  $N > 0$ . We proceed to show how they approximate  $\chi$  in the following result.

**Lemma 3.1.** — Assume that  $L > 2$ . Then  $\Phi_-(x) \leq \chi(x) \leq \Phi_+(x)$  for all  $x \in \mathbb{R}$ . Moreover, we have

$$\int_{-\infty}^{\infty} \Phi_{\pm}(x) dx = 1 \pm (2\Delta^{1/2} + \varepsilon_1). \quad (3.2)$$

*Proof.* — The equation (3.2) is easily proved as follows. We have

$$\begin{aligned} \int_{-\infty}^{\infty} \Phi_{\pm}(x) dx &= \int_{-1/2 \mp \Delta^{1/2}}^{1/2 \pm \Delta^{1/2}} \left( L \int_{-\infty}^{\infty} \Gamma((x - \mu)L) d\mu \right) dx \pm \varepsilon_1 \int_{-\infty}^{\infty} \Gamma(x) dx \\ &= \int_{-1/2 \mp \Delta^{1/2}}^{1/2 \pm \Delta^{1/2}} \left( \int_{-\infty}^{\infty} \Gamma(y) dy \right) d\mu \pm \varepsilon_1 \\ &= 1 \pm (2\Delta^{1/2} + \varepsilon_1), \end{aligned}$$

as claimed.

Let  $I = [-1/2, 1/2)$ . Next, we want to show that  $\chi(x) \leq \Phi_+(x)$  for all  $x \in \mathbb{R}$ . If  $x \notin I$  then obviously  $\Phi_+(x) > 0$ . If  $x \in I$  then

$$\begin{aligned} \Phi_+(x) &\geq L \int_{x - \Delta^{1/2}}^{x + \Delta^{1/2}} \Gamma((x - \mu)L) d\mu + \varepsilon_1 \exp\left(-\frac{\pi}{4}\right) \\ &= \int_{-L^{1/2}}^{L^{1/2}} \Gamma(-y) dy + \varepsilon_1 \exp\left(-\frac{\pi}{4}\right) \\ &= 1 - 2 \int_{L^{1/2}}^{\infty} \Gamma(y) dy + \varepsilon_1 \exp\left(-\frac{\pi}{4}\right) \\ &= 1 - \int_{\pi L}^{\infty} \frac{\exp(-z)}{\sqrt{\pi z}} dz + \varepsilon_1 \exp\left(-\frac{\pi}{4}\right) \\ &\geq 1 - \exp(-L) + \frac{\varepsilon_1}{3} = 1 + 7 \exp(-L) \geq 1. \end{aligned}$$

Hence we have established that  $\chi(x) \leq \Phi_+(x)$  for all  $x \in \mathbb{R}$ .

Finally we want to show that  $\Phi_-(x) \leq \chi(x)$  for all  $x \in \mathbb{R}$ . If  $x \in I$  then

$$\Phi_-(x) < L \int_{-\infty}^{\infty} \Gamma((x - \mu)L) d\mu = \int_{-\infty}^{\infty} \Gamma(y) dy = 1.$$



If  $1/2 \leq |x| \leq 1$  then

$$\begin{aligned}
\Phi_-(x) &\leq L \left( \int_{-\infty}^{\infty} - \int_{x-\Delta^{1/2}}^{x+\Delta^{1/2}} \right) \Gamma((x-\mu)L) d\mu - \varepsilon_1 \exp(-\pi) \\
&= 2 \int_{L^{1/2}}^{\infty} \Gamma(y) dy - \varepsilon_1 \exp(-\pi) \\
&= \int_{\pi L}^{\infty} \frac{\exp(-z)}{\sqrt{\pi z}} dz - \varepsilon_1 \exp(-\pi) \\
&\leq \exp(-L) - \frac{\varepsilon_1}{24} = 0.
\end{aligned}$$

If  $|x| > 1$  then for all  $\mu \in [-1/2 + \Delta^{1/2}, 1/2 - \Delta^{1/2}]$  we have

$$|x - \mu| \geq |x| - |\mu| > |x| - 1/2 > |x|/2.$$

Hence

$$\Phi_-(x) \leq L \Gamma\left(\frac{x}{2} \cdot L\right) \int_{-1/2+\Delta^{1/2}}^{1/2-\Delta^{1/2}} d\mu - \varepsilon_1 \Gamma(x) \leq L \exp\left(-\pi \frac{x^2 L^2}{4}\right) - \varepsilon_1 \exp(-\pi x^2).$$

By taking logarithms one easily verifies that the last line is  $\leq 0$  if  $|x| > 1$  and

$$\log \frac{L}{24} + L \leq \pi \left( \frac{L^2}{4} - 1 \right),$$

which is true for all  $L > 2$ . This completes the proof of  $\Phi_-(x) \leq \chi(x)$  for all  $x \in \mathbb{R}$  and therefore the proof of the lemma.  $\square$

From Lemma 3.1 one can easily deduce a similar result for the characteristic function of a general interval  $[a, b]$  by making the change of variables  $x \rightarrow (x - a)/(b - a) - 1/2$ , which maps the interval  $[a, b]$  bijectively onto  $[-1/2, 1/2]$ . We include this result here because it may be useful for future applications.

**Lemma 3.2.** — *Let  $a, b$  be real numbers with  $a < b$ . Denote the characteristic function of the interval  $[a, b]$  by  $\chi_{[a,b]}$ . Suppose that  $L > 2$  and set*

$$\Delta := L^{-1}, \quad \varepsilon_1 := 24 \exp(-L).$$

Define

$$\Phi_{\pm}^{a,b}(x) := L \int_{-1/2 \mp \Delta^{1/2}}^{1/2 \pm \Delta^{1/2}} \Gamma\left(\left(\frac{x-a}{b-a} - \frac{1}{2} - \mu\right)L\right) d\mu \pm \varepsilon_1 \Gamma\left(\frac{x-a}{b-a} - \frac{1}{2}\right).$$

Then we have  $\Phi_{\pm}^{a,b}(x) \leq \chi_{[a,b]}(x) \leq \Phi_{\pm}^{a,b}(x)$  for all  $x \in \mathbb{R}$  and

$$\int_{-\infty}^{\infty} \Phi_{\pm}^{a,b}(x) dx = \left(1 \pm (2\Delta^{1/2} + \varepsilon_1)\right) (b - a).$$

Moreover  $\Phi_{+}^{a,b}$  and  $\Phi_{-}^{a,b}$  are infinitely differentiable functions that have rapid decay at  $\pm\infty$ .

**3.2. Estimation of exponential integrals.** — Recall the definition (3.1) of the Gaussian function. We begin with a general upper bound for exponential integrals weighted by the Gaussian.

**Lemma 3.3.** — *Let  $f : \mathbb{R} \rightarrow \mathbb{R}$  be a smooth function and suppose that there exists  $\Lambda > 0$  and  $k \in \mathbb{N}$  such that  $|f^{(k)}(x)| \geq \Lambda$  for every  $x \in \mathbb{R}$ . Then we have*

$$\int_{-\infty}^{\infty} \Gamma(x) e(f(x)) dx \ll \Lambda^{-1/k}.$$

*Proof.* — It will be sufficient to prove the corresponding bound for the integral over  $[0, \infty)$  since the treatment of the integral over  $(-\infty, 0]$  is similar. For any  $a > 0$  it follows from integration by parts that

$$\int_0^a \Gamma(x) e(f(x)) dx = \Gamma(a) \int_0^a e(f(x)) dx - \int_0^a \Gamma'(y) \int_0^y e(f(x)) dx dy.$$

Now it follows from Lemma 8.10 in [18] and the hypotheses of the lemma that

$$\left| \int_0^y e(f(x)) dx \right| \leq k 2^k \Lambda^{-1/k},$$

for any  $y \leq a$ . Hence it readily follows that

$$\left| \int_0^a \Gamma(x) e(f(x)) dx \right| \leq k 2^k \Lambda^{-1/k} \left( \Gamma(a) + \int_0^a |\Gamma'(y)| dy \right) = k 2^k \Lambda^{-1/k}.$$

Taking the limit  $a \rightarrow \infty$  we are easily led to the desired bound. This completes the proof.  $\square$

The special case  $f(x) = -\beta x^3 - \alpha x$  will feature quite heavily in our work, corresponding to weighted Airy–Hardy integrals. For any  $\alpha, \beta \in \mathbb{R}$  such that  $\beta > 0$  we set

$$F(\alpha, \beta) := \int_{-\infty}^{\infty} \Gamma(x) e(-\beta x^3 - \alpha x) dx. \quad (3.3)$$

We begin by recording the trivial estimate

$$|F(\alpha, \beta)| \leq 1. \quad (3.4)$$

Furthermore, applying Lemma 3.3 with  $k = 3$ , we deduce that

$$F(\alpha, \beta) \ll \beta^{-1/3}. \quad (3.5)$$

Moreover, if  $\alpha > 0$ , then for all  $x \in \mathbb{R}$  we have

$$\frac{d(-\beta x^3 - \alpha x)}{dx} = -3\beta x^2 - \alpha \leq -\alpha < 0.$$

An application of Lemma 3.3 with  $k = 1$  now gives

$$F(\alpha, \beta) \ll \frac{1}{\alpha} \quad \text{for } \alpha > 0. \quad (3.6)$$

We henceforth assume that  $\alpha < 0$ . In this case we have stationary points which give a main term contribution. We write

$$\begin{aligned} F(\alpha, \beta) &= \int_0^{\infty} \Gamma(x) e(-\beta x^3 + |\alpha|x) dx + \int_0^{\infty} \Gamma(-x) e(\beta x^3 - |\alpha|x) dx \\ &= \overline{\int_0^{\infty} \Gamma(-x) e(\beta x^3 - |\alpha|x) dx} + \int_0^{\infty} \Gamma(-x) e(\beta x^3 - |\alpha|x) dx, \end{aligned} \quad (3.7)$$

where we take into account that our function  $\Gamma$  is even. We make a change of variables  $t = \beta^{1/3}x$ , getting

$$\int_0^\infty \Gamma(-x)e(\beta x^3 - |\alpha|x)dx = \beta^{-1/3} \int_0^\infty \Gamma(-\beta^{-1/3}t)e(t^3 - |\alpha|\beta^{-1/3}t)dt. \quad (3.8)$$

To estimate the integral on the right-hand side we use Theorem 2.2 in [17], which we record here for convenience.

**Lemma 3.4 (Stationary phase with weights).** — *Let  $f(z)$ ,  $g(z)$  be two functions of the complex variable  $z$  and  $[a, b]$  be a real interval such that the following hold.*

- (a) *For  $a \leq x \leq b$  the function  $f(x)$  is real and  $f''(x) > 0$ .*
- (b) *For a certain positive differentiable function  $\mu(x)$ , defined on  $[a, b]$ ,  $f(z)$  and  $g(z)$  are analytic for  $a \leq x \leq b$ ,  $|z - x| \leq \mu(x)$ .*
- (c) *There exist positive functions  $F(x)$ ,  $G(x)$  defined on  $[a, b]$  such that for  $a \leq x \leq b$ ,  $|z - x| \leq \mu(x)$  we have*

$$g(z) \ll G(x), \quad f'(z) \ll \frac{F(x)}{\mu(x)}, \quad |f''(z)|^{-1} \ll \frac{\mu(x)^2}{F(x)},$$

and the implied constants are absolute.

Let  $k \in \mathbb{R}$  and if  $f'(x) + k$  has a zero in  $[a, b]$  denote it by  $x_0$ . Let the values of  $f(x)$ ,  $g(x)$  and so on, at  $a$ ,  $x_0$ , and  $b$  be characterised by the suffixes  $a$ ,  $0$  and  $b$ , respectively. Then, for some absolute constant  $C > 0$ , we have

$$\begin{aligned} \int_a^b g(x)e(f(x) + kx) dx &= \frac{g_0}{\sqrt{f_0''}} \cdot e\left(f_0 + kx_0 + \frac{1}{8}\right) + O\left(G_0\mu_0F_0^{-3/2}\right) \\ &\quad + O\left(\int_a^b G(x) \exp(-C|k|\mu(x) - CF(x)) (dx + |d\mu(x)|)\right) \\ &\quad + O\left(G_a\left(|f'_a + k| + \sqrt{f''_a}\right)^{-1} + G_b\left(|f'_b + k| + \sqrt{f''_b}\right)^{-1}\right). \end{aligned} \quad (3.9)$$

If  $f'(x) + k$  has no zero in  $[a, b]$ , then the terms involving  $x_0$  are to be omitted.

We apply Lemma 3.4 with  $0 < a < b$  and

$$\begin{aligned} f(z) &= z^3, \quad g(z) = \exp(-\pi\beta^{-2/3}z^2), \quad \mu(x) = x/2, \quad F(x) = x^3, \\ G(x) &= \exp(-\pi\beta^{-2/3}x^2/2), \quad k = -|\alpha|\beta^{-1/3}, \end{aligned}$$

and let  $a$  tend to 0 and  $b$  to  $\infty$ . Here we note that the choice of  $G(x)$  above is possible since

$$g(z) \ll \exp\left(-\pi\beta^{-2/3}x^2/2\right)$$

if  $z \in \mathbb{C}$  with  $|z - x| \leq x/2$ . From (3.8) and (3.9) we deduce that

$$\begin{aligned} \int_0^\infty \Gamma(-x)e(\beta x^3 - |\alpha|x)dx &= \frac{\exp(-|\pi\alpha|/(3\beta))}{(12|\alpha|\beta)^{1/4}} \cdot e\left(\frac{1}{8} - \frac{2|\alpha|^{3/2}}{3^{3/2}\beta^{1/2}}\right) \\ &\quad + O\left(\frac{1}{|\alpha|} + \frac{\beta^{1/4}}{|\alpha|^{7/4}}\right). \end{aligned} \quad (3.10)$$

To see this we note that the stationary point is

$$x_0 = 3^{-1/2}|\alpha|^{1/2}\beta^{-1/6}. \quad (3.11)$$

Using this, we easily calculate the main term on the right-hand side of (3.10). Then we simplify the integral on the right-hand side of (3.9) by removing the terms  $\exp(-CF(x))$  and  $G(x)$  (dropping these terms can only make the integral larger) and observing that  $|\mathrm{d}\mu(x)|$  is the same as  $\mathrm{d}x/2$ . What remains in the integrand is the term  $\exp(-Ck\mu(x)) = \exp(-Ckx/2)$ . The integral of this term is  $\ll 1/|k|$ . Plugging in our value of  $k$  and multiplying by the factor  $\beta^{-1/3}$ , we obtain the term  $1/|\alpha|$  in the  $O$ -term on the right-hand side of (3.10). Furthermore, we calculate the term  $G_0\mu_0F_0^{-3/2}$  on the right-hand side of (3.9) using (3.11). Multiplying the result with  $\beta^{-1/3}$ , we get the term

$$\exp(-|\pi\alpha|/(6\beta)) \cdot \frac{\beta^{1/4}}{|\alpha|^{7/4}} \ll \frac{\beta^{1/4}}{|\alpha|^{7/4}}$$

on the right-hand side of (3.10). The term  $G_a(|f'_a + k| + \sqrt{f''_a})^{-1}$  on the right-hand side of (3.9) tends to  $1/|k|$  as  $a$  tends to 0, which is the value with which we bounded the integral on the right-hand side of (3.9). Finally, the term  $G_b(|f'_b + k| + \sqrt{f''_b})^{-1}$  tends to 0 as  $b$  tends to infinity. Combining everything we obtain (3.10).

From (3.7), (3.8) and (3.10), we get

$$F(\alpha, \beta) = \frac{2^{1/2} \exp(-|\pi\alpha|/(3\beta))}{(3|\alpha|\beta)^{1/4}} \cdot \cos\left(2\pi\left(\frac{1}{8} - \frac{2|\alpha|^{3/2}}{3^{3/2}\beta^{1/2}}\right)\right) + O\left(\frac{1}{|\alpha|} + \frac{\beta^{1/4}}{|\alpha|^{7/4}}\right),$$

whence

$$F(\alpha, \beta) = \frac{2^{1/2} \exp(-|\pi\alpha|/(3\beta))}{(3|\alpha|\beta)^{1/4}} \cdot \cos\left(2\pi\left(\frac{1}{8} - \frac{2|\alpha|^{3/2}}{3^{3/2}\beta^{1/2}}\right)\right) + O\left(\frac{1}{|\alpha|}\right). \quad (3.12)$$

Indeed, if  $\alpha < -\beta^{1/3}$  then the estimate is obvious. If instead  $-\beta^{1/3} \leq \alpha < 0$  then the main term is dominated by the error term and the estimate  $F(\alpha, \beta) = O(1/|\alpha|)$  follows from (3.5). Bringing (3.12) together with (3.6) we may now record the upper bound

$$F(\alpha, \beta) \ll \frac{\exp(-|\pi\alpha|/(3\beta))}{(|\alpha|\beta)^{1/4}} + \frac{1}{|\alpha|} \quad \text{for any } \alpha \neq 0. \quad (3.13)$$

We will need yet another bound that is good for large  $|\alpha|$ . To this end, we take into account that  $\Gamma$  extends to the entire function  $\Gamma(z)$  for  $z \in \mathbb{C}$  and use complex analysis. Shifting the line of integration, we get

$$\begin{aligned} F(\alpha, \beta) &= \int_{-\infty-ic}^{\infty-ic} \Gamma(s) e(-\beta s^3 - \alpha s) \mathrm{d}s \\ &= \int_{-\infty}^{\infty} \Gamma(x - ic) e(-\beta(x - ic)^3 - \alpha(x - ic)) \mathrm{d}x \\ &= \exp(\pi c^2 - 2\pi\alpha c + 2\pi\beta c^3) \int_{-\infty}^{\infty} \exp(-\pi x^2 - 6\pi\beta c x^2) e(cx - \beta x^3 + 3\beta c^2 x - \alpha x) \mathrm{d}x \\ &\ll \exp(\pi c^2 - 2\pi\alpha c + 2\pi\beta c^3) \int_{-\infty}^{\infty} \exp(-\pi(1 + 6\beta c)x^2) \mathrm{d}x, \end{aligned}$$

which converges if  $c > -1/(6\beta)$ . We choose  $c := \text{sign}(\alpha)/(12\beta)$ , getting

$$F(\alpha, \beta) \ll \exp\left(\frac{\pi}{12\beta^2} - \frac{\pi|\alpha|}{6\beta}\right) \ll \exp\left(-\pi \cdot \frac{|\alpha|}{12\beta}\right), \quad (3.14)$$

if  $|\alpha| \geq 1/\beta$ .

**3.3. Arithmetic functions.** — In this section we collect together some of the arithmetic functions that feature in our work and some of their basic properties. For any  $a \in \mathbb{R}$  we set

$$\sigma_a(n) := \sum_{d|n} d^a.$$

The cases  $a = -1/2$  and  $a = -1$  will arise quite often in our analysis. Let  $\omega(n)$  denote the number of distinct prime factors of  $n$ . Then it is straightforward to show that

$$\sigma_{-1/2}(n) \ll (1 + \varepsilon)^{\omega(n)},$$

for any  $\varepsilon > 0$ . Likewise we have

$$\sigma_{-1}(n) \ll \log \log n \ll (\log n)^\varepsilon.$$

Next let

$$\varphi^*(n) := \frac{\varphi(n)}{n} = \prod_{p|n} \left(1 - \frac{1}{p}\right).$$

We observe that  $\varphi^*(ab)\varphi^*(a, b) = \varphi^*(a)\varphi^*(b)$ , where we write  $\varphi^*(a, b) = \varphi^*((a, b))$  for short. It is then an elementary exercise to check that

$$\sum_{\substack{k|n \\ (k, a)=1}} \frac{\mu(k)}{k} = \frac{\varphi^*(n)}{\varphi^*(a, n)}, \quad \sum_{\substack{k|n \\ (k, a)=1}} \frac{\mu(k)}{k\varphi^*(bk)} = \frac{\varphi^*(n, ab)}{\varphi^*(bn)\varphi^*(a, b, n)} \prod_{\substack{p|n \\ p \nmid ab}} \left(1 - \frac{2}{p}\right), \quad (3.15)$$

for any  $a, b \in \mathbb{N}$ .

Let  $z \in \mathbb{R}_{\geq 0}$  and  $\theta \in [0, 1)$ . Then we have the estimates

$$\sum_{n \leq x} \frac{z^{\omega(n)}}{n^\theta} \ll x^{1-\theta} (\log x)^{z-1}, \quad \sum_{n \leq x} \frac{z^{\omega(n)}}{n} \ll (\log x)^z, \quad (3.16)$$

following from [23, § II.6] and partial summation. Next we record the bound

$$\sum_{1 \leq n \leq x} (n, r)^{1/2} \leq \sum_{d|r} \sum_{\substack{1 \leq n \leq x \\ d|n}} d^{1/2} \leq x \sum_{d|r} d^{-1/2} = x \sigma_{-1/2}(r) \ll (1 + \varepsilon)^{\omega(r)} x, \quad (3.17)$$

for any  $r \in \mathbb{N}$ . Using partial summation and the fact that the function  $\Gamma$ , defined in (3.1), has rapid decay, it follows that

$$\sum_{n \in \mathbb{N}} \Gamma\left(\frac{n}{x}\right) (n, r)^{1/2} \ll (1 + \varepsilon)^{\omega(r)} x. \quad (3.18)$$

Finally we record the estimate

$$\sum_{n \leq x} \frac{z^{\omega(n)} \sigma_1(n)}{n^2} \leq \sum_{kn' \leq x} \frac{z^{\omega(k) + \omega(n')}}{n'^2 k} \ll (\log x)^z. \quad (3.19)$$

We will make frequent use of the estimates in this section with little further comment.

#### 4. Exponential sums

In this section we collect together some facts concerning complete exponential sums that will be required in our work. We begin with a general upper bound. For a polynomial  $f \in \mathbb{Z}[X]$  of degree  $n$  with precisely  $m$  distinct roots  $\zeta_1, \dots, \zeta_m$  and factorisation

$$f(X) = A(X - \zeta_1)^{\eta_1} \cdots (X - \zeta_m)^{\eta_m},$$

define the semi-discriminant of  $f$  to be

$$\Delta(f) := A^{2n-2} \prod_{i \neq j} (\zeta_i - \zeta_j)^{\eta_i \eta_j}$$

and the exponent of  $f$  to be

$$\eta(f) := \max\{\eta_1, \dots, \eta_m\}.$$

With this notation in mind we have the following result.

**Lemma 4.1.** — *Let  $Q \in \mathbb{N}$  and let  $g$  be a polynomial over  $\mathbb{Z}$  with degree  $n + 1$  for  $n \geq 2$ . Let  $\Delta = \Delta(g')$  and  $\eta = \eta(g')$ , where  $g'$  is the derivative of  $g$ . Then we have*

$$\left| \sum_{x \bmod Q} e\left(\frac{g(x)}{Q}\right) \right| \leq Q^{1-1/(2\eta)} (\Delta, Q)^{1/(2\eta)} n^{\omega(Q)}.$$

*Proof.* — A version of this result with  $n^{\omega(Q)}$  replaced by  $\tau_n(Q)$  is the principal result in work of Loxton and Smith [20]. An inspection of the proof reveals that the sharper version recorded here is also true.  $\square$

Our work will hinge upon a decent understanding of the exponential sum

$$E(c, d; q) := \sum_{z \bmod q}^* e\left(\frac{cz^3 - dz^2}{q}\right) \quad (4.1)$$

for  $c, d, q \in \mathbb{Z}$  such that  $q > 0$ . Here the asterisk denotes a summation in which  $z$  runs over residue classes modulo  $q$  that are coprime to  $q$ . The rest of this section is dedicated to its explicit evaluation as far as this is possible. In fact it will suffice to do so for prime power moduli  $q = p^t$  since the sums satisfy a multiplicativity property, as recorded in the following result.

**Lemma 4.2.** — *Let  $c, d \in \mathbb{Z}$  and assume that  $q_1, \dots, q_r$  are pairwise coprime. Then we have*

$$E(c, d; q_1 \cdots q_r) = \prod_{j=1}^r E(c\bar{q}_j, d\bar{q}_j; q_j), \quad (4.2)$$

where  $\tilde{q}_j = q_1 \cdots q_r / q_j$  and  $\bar{q}_j \tilde{q}_j \equiv 1 \pmod{q_j}$ .

*Proof.* — We prove (4.2) only for  $r = 2$ . By induction, the result can then be easily extended to general  $r$ . If  $(q_1, q_2) = 1$  then

$$\begin{aligned} E(c, d; q_1 q_2) &= \sum_{z \bmod q_1 q_2}^* e\left(\frac{cz^3 - dz^2}{q_1 q_2}\right) \\ &= \sum_{x_1 \bmod q_1}^* \sum_{x_2 \bmod q_2}^* e\left(\frac{c(x_1 q_2 + x_2 q_1)^3 - d(x_1 q_2 + x_2 q_1)^2}{q_1 q_2}\right) \\ &= \sum_{x_1 \bmod q_1}^* e\left(\frac{cq_2^2 x_1^3 - dq_2 x_1^2}{q_1}\right) \sum_{x_2 \bmod q_2}^* e\left(\frac{cq_1^2 x_2^3 - dq_1 x_2^2}{q_2}\right). \end{aligned}$$

Making the change of variables  $x_1 = \bar{q}_2 y_1$ ,  $x_2 = \bar{q}_1 y_2$  with  $\bar{q}_1 q_1 \equiv 1 \pmod{q_2}$  and  $\bar{q}_2 q_2 \equiv 1 \pmod{q_1}$ , we deduce that

$$E(c, d; q_1 q_2) = E(c\bar{q}_2, d\bar{q}_2; q_1) E(c\bar{q}_1, d\bar{q}_1; q_2). \quad (4.3)$$

This completes the proof.  $\square$

It shall transpire that the evaluation of  $E(c, d; p^t)$  for  $p = 2, 3$  is more complicated than for primes  $p > 3$ . We begin by making a useful observation concerning these bad primes.

**Lemma 4.3.** — Suppose that  $4 \mid q$  or  $9 \mid q$ . Then we have

$$E(c, d; q) \neq 0 \Rightarrow \begin{cases} 2 \mid c & \text{if } 4 \mid q, \\ 3 \mid d & \text{if } 9 \mid q. \end{cases}$$

*Proof.* — In view of Lemma 4.2 it will suffice to show that

$$E(c, d; p^t) \neq 0 \Rightarrow \begin{cases} 2 \mid c & \text{if } p = 2, \\ 3 \mid d & \text{if } p = 3, \end{cases}$$

if  $t \geq 2$ . But clearly

$$\begin{aligned} E(c, d; p^t) &= \sum_{x=1}^{p^{t-1}} \sum_{y=1}^p e \left( \frac{c(x + yp^{t-1})^3 - d(x + yp^{t-1})^2}{p^t} \right) \\ &= \sum_{x=1}^{p^{t-1}} e \left( \frac{cx^3 - dx^2}{p^t} \right) \sum_{y=1}^p e \left( \frac{3cx^2 - 2dx}{p} \cdot y \right), \end{aligned}$$

and if  $p = 2$  (resp.  $p = 3$ ) the inner sum in the last line equals 0 unless  $2 \mid c$  (resp.  $3 \mid d$ ).  $\square$

For  $q \in \mathbb{N}$ , we set

$$w_2(q) := \begin{cases} 1 & \text{if } 4 \mid q, \\ 0 & \text{otherwise,} \end{cases} \quad (4.4)$$

and

$$w_3(q) := \begin{cases} 1 & \text{if } 9 \mid q, \\ 0 & \text{otherwise.} \end{cases} \quad (4.5)$$

By Lemma 4.3 it will suffice to study exponential sums of the form

$$E_{f_2, f_3}(c, d; q) := E(2^{f_2}c, 3^{f_3}d; q),$$

where

$$w_2(q) \leq f_2 \leq 1 \quad \text{and} \quad w_3(q) \leq f_3 \leq 1. \quad (4.6)$$

We will assume (4.6) throughout the sequel. We note that equation (4.3) translates into

$$E_{f_2, f_3}(c, d; q_1 q_2) = E_{f_2, f_3}(c \overline{q_2}, d \overline{q_2}; q_1) E_{f_2, f_3}(c \overline{q_1}, d \overline{q_1}; q_2). \quad (4.7)$$

**4.1. Prime power moduli.** — In the following we evaluate the exponential sum  $E_{f_2, f_3}(c, d; q)$  for prime power moduli  $q = p^t$ . We set

$$r(p) := \begin{cases} 6 & \text{if } p = 2, \\ 5 & \text{if } p = 3, \\ 2 & \text{otherwise.} \end{cases} \quad (4.8)$$

We shall treat  $E_{f_2, f_3}(c, d; p^t)$  differently for the cases  $t < r(p)$  and  $t \geq r(p)$ .

*Case 1:*  $1 \leq t < r(p)$ . — For  $p = 2, 3$ , we shall simply use the trivial estimate

$$|E_{f_2, f_3}(c, d; p^t)| \leq 100 \quad \text{if } p = 2, 3 \text{ and } t < r(p).$$

Assume now that  $p > 3$  and  $t < r(p)$ , in which case we have  $t = 1$ . If  $p \mid c$  and  $p \mid d$ , then we have

$$E_{f_2, f_3}(c, d; p) = E_{f_2, f_3}(0, 0; p) = \varphi(p).$$

If  $p \mid c$  and  $p \nmid d$ , then  $E_{f_2, f_3}(c, d; p)$  is a quadratic Gauss sum

$$\begin{aligned} E_{f_2, f_3}(c, d; p) &= E_{f_2, f_3}(0, d; p) = \sum_{z=1}^{p-1} e\left(-\frac{3^{f_3} dz^2}{p}\right) = \sum_{z=1}^p e\left(-\frac{3^{f_3} dz^2}{p}\right) - 1 \\ &= \sqrt{p} \left(\frac{-3^{f_3} d}{p}\right) \epsilon_p - 1, \end{aligned}$$

where  $\left(\frac{\cdot}{p}\right)$  is the Legendre symbol and

$$\epsilon_q := \begin{cases} 1 & \text{if } q \equiv 1 \pmod{4}, \\ i & \text{if } q \equiv -1 \pmod{4}. \end{cases} \quad (4.9)$$

If  $p \nmid c$ , then there is no simple closed expression for  $E_{f_2, f_3}(c, d; p)$ , but by [24] we have the Weil bound

$$|E_{f_2, f_3}(c, d; p)| \leq 2\sqrt{p}.$$

*Case 2:*  $t \geq r(p)$ . — Here we first prove the following result.

**Lemma 4.4.** — *Let  $p$  be a prime and let  $t \geq r(p) = r$ . Then we have the following.*

(i) *If  $p^{t-r+2} \nmid c$  and  $p^{t-r+2} \nmid d$ , then*

$$E_{f_2, f_3}(c, d; p^t) \neq 0 \Rightarrow (c, p^t) = p^s = (d, p^t) \text{ for some } s.$$

*In this case, we have*

$$E_{f_2, f_3}(c, d; p^t) = p^s E_{f_2, f_3}\left(\frac{c}{p^s}, \frac{d}{p^s}; p^{t-s}\right). \quad (4.10)$$

(ii) *If  $p^{t-r+2} \mid c$  or  $p^{t-r+2} \mid d$ , then*

$$E_{f_2, f_3}(c, d; p^t) \neq 0 \Rightarrow p^{t-r+1} \mid (c, d).$$

*In this case, we have*

$$E_{f_2, f_3}(c, d; p^t) = p^{t-r+1} E_{f_2, f_3}\left(\frac{c}{p^{t-r+1}}, \frac{d}{p^{t-r+1}}; p^{r-1}\right).$$

*Proof.* — We recall that we assume (4.6) holds, where here  $q = p^t$ . Hence, under the conditions of this lemma, we have  $f_2 = 1$  if  $p = 2$  and  $f_3 = 1$  if  $p = 3$ .

Our starting point is the following computation, valid for all primes  $p$ . Set

$$\nu = \nu(p) := \begin{cases} 1 & \text{if } p = 2, 3, \\ 0 & \text{otherwise.} \end{cases}$$



Then

$$\begin{aligned}
E_{f_2, f_3}(c, d; p^t) &= \sum_{x=1}^{p^{t-1-\nu}}^* \sum_{y=1}^{p^{1+\nu}} e\left(\frac{2^{f_2} c(x + yp^{t-1-\nu})^3 - 3^{f_3} d(x + yp^{t-1-\nu})^2}{p^t}\right) \\
&= \sum_{x=1}^{p^{t-1-\nu}}^* e\left(\frac{2^{f_2} cx^3 - 3^{f_3} dx^2}{p^t}\right) \sum_{y=1}^{p^{1+\nu}} e\left(\frac{3 \cdot 2^{f_2} cx^2 - 2 \cdot 3^{f_3} dx}{p^{1+\nu}} \cdot y\right) \\
&= p^{1+\nu} \sum_{\substack{x=1 \\ 3 \cdot 2^{f_2} cx \equiv 2 \cdot 3^{f_3} d \pmod{p^{1+\nu}}}}^{p^{t-1-\nu}} e\left(\frac{2^{f_2} cx^3 - 3^{f_3} dx^2}{p^t}\right) \\
&= p^{1+\nu} \sum_{\substack{x=1 \\ 3^{1-f_3} cx \equiv 2^{1-f_2} d \pmod{p}}}^{p^{t-1-\nu}} e\left(\frac{2^{f_2} cx^3 - 3^{f_3} dx^2}{p^t}\right).
\end{aligned}$$

We observe that this equals 0 if  $(c, p) \neq (d, p)$  because in this case the congruence

$$3^{1-f_3} cx \equiv 2^{1-f_2} d \pmod{p}$$

is not solvable for  $x$  coprime to  $p$ . Moreover, if  $(c, p) = p = (d, p)$ , then we have

$$E_{f_2, f_3}(c, d; p^t) = p E_{f_2, f_3}\left(\frac{c}{p}, \frac{d}{p}; p^{t-1}\right). \quad (4.11)$$

Using this observation, we prove Lemma 4.4 by induction over  $t$ . The base case  $t = r$  is a trivial consequence of our work so far, where, for the proof of (4.10), it is important to note that  $t > s$  by the assumptions in (i).

Now assume that  $t > r$  and that the assertion has been established for the exponent  $t-1$ . First, let the conditions  $p^{t-r+2} \nmid c$  and  $p^{t-r+2} \nmid d$  in case (i) be satisfied and suppose that  $E_{f_2, f_3}(c, d; p^t) \neq 0$ . Then, by the afore-mentioned observation, we have  $(c, p^t) = 1$  if and only if  $(d, p^t) = 1$ . If  $(c, p^t) > 1$  and  $(d, p^t) > 1$ , then from (4.11) and the induction hypothesis, it follows that  $(c/p, p^{t-1}) = p^{s-1} = (d/p, p^{t-1})$  and hence  $(c, p^t) = p^s = (d, p^t)$  for some  $s$ , and

$$E_{f_2, f_3}(c, d; p^t) = p E_{f_2, f_3}\left(\frac{c}{p}, \frac{d}{p}; p^{t-1}\right) = p^s E_{f_2, f_3}\left(\frac{c}{p^s}, \frac{d}{p^s}; p^{t-s}\right),$$

as claimed.

For case (ii) we suppose that  $p^{t-r+2} \mid c$  and  $E_{f_2, f_3}(c, d; p^t) \neq 0$ . Then, by our observation above, we have  $(d, p^t) > 1$ , and from (4.11) and the induction hypothesis, it follows that  $p^{t-r} \mid (d/p)$  and hence  $p^{t-r+1} \mid d$ . Thus

$$E_{f_2, f_3}(c, d; p^t) = p E_{f_2, f_3}\left(\frac{c}{p}, \frac{d}{p}; p^{t-1}\right) = p^{t-r+1} E_{f_2, f_3}\left(\frac{c}{p^{t-r+1}}, \frac{d}{p^{t-r+1}}; p^{r-1}\right),$$

as claimed. The case  $p^{t-r+2} \mid d$  is handled similarly.  $\square$

Let  $t \geq r(p) = r$ . By Lemma 4.4 we have

$$E_{f_2, f_3}(c, d; p^t) = p^{t-r+1} E_{f_2, f_3}\left(\frac{c}{p^{t-r+1}}, \frac{d}{p^{t-r+1}}; p^{r-1}\right)$$

if  $p^{t-r+1} \mid (c, d)$ , and the exponential sum on the right-hand side has been dealt with before. If  $p^{t-r+1} \nmid (c, d)$ , then by the same Lemma 4.4, we have  $(c, p^t) = p^s = (d, p^t)$  for some  $s \leq t - r$ . The

following result gives an explicit evaluation of  $E_{f_2, f_3}(c, d; p^t)$  in this situation. Before stating the lemma, we define a residue symbol

$$\left\{ \frac{\cdot}{2^u} \right\} := \begin{cases} \frac{1}{\sqrt{2}}(1 + e(\cdot/4)) & \text{if } 2 \nmid u, \\ e(\cdot/8) & \text{if } 2 \mid u. \end{cases} \quad (4.12)$$

for  $u \geq 3$ . Furthermore for any odd integer  $n$  we let  $(\frac{\cdot}{n})$  denote the Jacobi symbol. Throughout the following, in place of  $\bar{x}$ , we often use the notation  $x^{-1}$  for the multiplicative inverse. We now have the following result.

**Lemma 4.5.** — *Let  $p$  be a prime and  $t \geq r(p) = r$ . Suppose that  $(c, p^t) = p^s = (d, p^t)$  and  $s \leq t - r$ . Set  $c_1 = c/p^s$ ,  $d_1 = d/p^s$  and  $u = t - s$ . Suppose that*

$$A \equiv 4^{1-f_2} \cdot 27^{f_3-1} \pmod{p^u}.$$

Then

$$E_{f_2, f_3}(c, d; p^t) = p^{t-u/2} e\left(\frac{-A \bar{c}_1^{-2} d_1^3}{p^u}\right) R(f_3, d_1, p^u), \quad (4.13)$$

where

$$R(f_3, d_1, p^u) := \begin{cases} \left(\frac{3^{f_3} d_1}{p^u}\right) \epsilon_{p^u} & \text{if } p > 3, \\ \sqrt{3} \left(\frac{d_1}{3^{u-1}}\right) \epsilon_{3^{u-1}} & \text{if } p = 3, \\ \sqrt{2} \left\{ \frac{3^{f_3} d_1}{2^u} \right\} & \text{if } p = 2. \end{cases}$$

*Proof.* — By (4.10), we have

$$E_{f_2, f_3}(c, d; p^t) = p^s E_{f_2, f_3}(c_1, d_1; p^u) = p^s E(p^u), \quad (4.14)$$

say. We further observe that

$$(c_1, p) = 1 = (d_1, p), \quad u \geq r.$$

Now it remains to consider  $E(p^u)$ .

If  $u$  is even then we set  $u = 2v$  and if  $u$  is odd then we set  $u = 2v + 1$ . Note that  $v \geq 1$  since  $u \geq 2$ . The starting point of our proof are the identities

$$\begin{aligned} E(p^u) &= \sum_{x \bmod p^v}^* \sum_{y \bmod p^v} e\left(\frac{2^{f_2} c_1 (x + yp^v)^3 - 3^{f_3} d_1 (x + yp^v)^2}{p^{2v}}\right) \\ &= \sum_{x \bmod p^v}^* e\left(\frac{2^{f_2} c_1 x^3 - 3^{f_3} d_1 x^2}{p^{2v}}\right) \sum_{y \bmod p^v} e\left(\frac{3 \cdot 2^{f_2} c_1 x^2 - 2 \cdot 3^{f_3} d_1 x}{p^v} \cdot y\right) \\ &= p^v \sum_{\substack{x \bmod p^v \\ 3 \cdot 2^{f_2} c_1 x \equiv 2 \cdot 3^{f_3} d_1 \pmod{p^v}}}^* e\left(\frac{2^{f_2} c_1 x^3 - 3^{f_3} d_1 x^2}{p^{2v}}\right) \end{aligned} \quad (4.15)$$

for  $u$  even, and

$$\begin{aligned} E(p^u) &= \sum_{x \bmod p^{v+1}}^* \sum_{y \bmod p^v} e\left(\frac{2^{f_2} c_1 (x + yp^{v+1})^3 - 3^{f_3} d_1 (x + yp^{v+1})^2}{p^{2v+1}}\right) \\ &= \sum_{x \bmod p^{v+1}}^* e\left(\frac{2^{f_2} c_1 x^3 - 3^{f_3} d_1 x^2}{p^{2v+1}}\right) \sum_{y \bmod p^v} e\left(\frac{3 \cdot 2^{f_2} c_1 x^2 - 2 \cdot 3^{f_3} d_1 x}{p^v} \cdot y\right) \\ &= p^v \sum_{\substack{x \bmod p^{v+1} \\ 3 \cdot 2^{f_2} c_1 x \equiv 2 \cdot 3^{f_3} d_1 \pmod{p^v}}}^* e\left(\frac{2^{f_2} c_1 x^3 - 3^{f_3} d_1 x^2}{p^{2v+1}}\right) \end{aligned} \quad (4.16)$$

for  $u$  odd. In the following we treat the cases  $p > 3$ ,  $p = 3$ ,  $p = 2$ ,  $u$  even,  $u$  odd separately.

*Case I:  $p > 3$ .* — For convenience, we use the notations

$$m = 2^{f_2} c_1, \quad n = 3^{f_3} d_1. \quad (4.17)$$

Suppose first that  $u$  is even. From (4.15), it follows that

$$E(p^u) = p^v \sum_{\substack{x \bmod p^v \\ 3mx \equiv 2n \bmod p^v}}^* e\left(\frac{mx^3 - nx^2}{p^{2v}}\right).$$

The congruence in the last line has precisely one solution modulo  $p^v$ , and using Hensel's lemma, this solution can be lifted uniquely to a solution of  $3mx \equiv 2n \bmod p^{2v}$ , given by

$$x \equiv \overline{3m} \cdot 2n \bmod p^{2v},$$

where  $\overline{3m}$  is the multiplicative inverse of  $3m$  modulo  $p^{2v}$ . Hence the sum collapses into a single term

$$\begin{aligned} e\left(\frac{mx^3 - nx^2}{p^{2v}}\right) &= e\left(\frac{m(\overline{3m} \cdot 2n)^3 - n(\overline{3m} \cdot 2n)^2}{p^{2v}}\right) = e\left(\frac{(8 \cdot \overline{27} - 4 \cdot \overline{9}) \overline{m}^2 n^3}{p^{2v}}\right) \\ &= e\left(\frac{-4 \cdot \overline{27} \cdot \overline{m}^2 n^3}{p^{2v}}\right), \end{aligned}$$

whence

$$E(p^u) = p^{u/2} e\left(\frac{-4 \cdot \overline{27} \cdot \overline{m}^2 n^3}{p^u}\right). \quad (4.18)$$

Next suppose that  $u$  is odd. From (4.16), it follows that

$$\begin{aligned} E(p^u) &= p^v \sum_{\substack{x \bmod p^{v+1} \\ 3mx \equiv 2n \bmod p^v}}^* e\left(\frac{mx^3 - nx^2}{p^{2v+1}}\right) \\ &= p^v \sum_{h=1}^p e\left(\frac{m(x + hp^v)^3 - n(x + hp^v)^2}{p^{2v+1}}\right) \\ &= p^v e\left(\frac{mx^3 - nx^2}{p^{2v+1}}\right) \sum_{h=1}^p e\left(\frac{3mx^2 hp^v + 3mxh^2 p^{2v} - 2nxh p^v - nh^2 p^{2v}}{p^{2v+1}}\right), \end{aligned} \quad (4.19)$$

where in the last two lines  $x$  satisfies the congruence  $3mx \equiv 2n \bmod p^v$ . Again this solution can be lifted uniquely to a solution

$$x \equiv \overline{3m} \cdot 2n \bmod p^{2v+1},$$

where  $\overline{3m}$  is the multiplicative inverse of  $3m$  modulo  $p^{2v+1}$ . Now, similarly as in the case of even  $u$ , the numerator in the first exponential in the last line of (4.19) takes the form  $mx^3 - nx^2 \equiv -4 \cdot \overline{27} \cdot \overline{m}^2 n^3 \bmod p^{2v+1}$ . Moreover, we have  $3mx^2 \equiv \overline{3m} \cdot 4n^2 \bmod p^{2v+1}$ ,  $2nx \equiv \overline{3m} \cdot 4n^2 \bmod p^{2v+1}$  and  $3mx \equiv 2n \bmod p^{2v+1}$ . Therefore, the numerator in the second exponential in the last line of (4.19) simplifies into  $nh^2 p^{2v}$ . It follows that

$$E(p^u) = p^v e\left(\frac{-4 \cdot \overline{27} \cdot \overline{m}^2 n^3}{p^{2v+1}}\right) \sum_{h=1}^p e\left(\frac{nh^2}{p}\right).$$

Evaluating the quadratic Gauss sum on the right-hand side, we obtain

$$E(p^u) = p^{v+1/2} e\left(\frac{-4 \cdot \overline{27} \cdot \overline{m}^2 n^3}{p^{2v+1}}\right) \left(\frac{n}{p}\right) \epsilon_p = p^{u/2} e\left(\frac{-4 \cdot \overline{27} \cdot \overline{m}^2 n^3}{p^u}\right) \left(\frac{n}{p}\right) \epsilon_p.$$

Combining this with (4.18) into a single equation, we deduce that

$$E(p^u) = p^{u/2} e \left( \frac{-4 \cdot \overline{27} \cdot \overline{m}^2 n^3}{p^u} \right) \left( \frac{n}{p^u} \right) \epsilon_{p^u}, \quad (4.20)$$

where  $\epsilon_{p^u}$  is defined as in (4.9). Combining (4.14), (4.17) and (4.20), we obtain (4.13) for  $p > 3$ .

*Case II:  $p = 3$ .* — We note that  $f_3 = 1$  by (4.6) and use the notations

$$m := 2^{f_2} c_1, \quad n := d_1. \quad (4.21)$$

Suppose first that  $u$  is even. Note that  $v \geq 3$  since  $u \geq 6$ . From (4.15) it follows that

$$\begin{aligned} E(3^u) &= 3^v \sum_{\substack{x \bmod 3^v \\ mx \equiv 2n \bmod 3^{v-1}}}^* e \left( \frac{mx^3 - 3nx^2}{3^{2v}} \right) \\ &= 3^v \sum_{h=1}^3 e \left( \frac{m(x + h \cdot 3^{v-1})^3 - 3n(x + h \cdot 3^{v-1})^2}{3^{2v}} \right) \\ &= 3^v e \left( \frac{mx^3 - 3nx^2}{3^{2v}} \right) \\ &\quad \times \sum_{h=1}^3 e \left( \frac{mx^2 h \cdot 3^v + mxh^2 \cdot 3^{2v-1} - 2n x h \cdot 3^v - nh^2 \cdot 3^{2v-1}}{3^{2v}} \right), \end{aligned} \quad (4.22)$$

where in the last two lines  $x$  satisfies the congruence  $mx \equiv 2n \bmod 3^{v-1}$ . The only solution of this congruence can be lifted uniquely to a solution of  $mx \equiv 2n \bmod 3^{2v}$ , given by

$$x \equiv \overline{m} \cdot 2n \bmod 3^{2v},$$

where  $\overline{m}$  is the multiplicative inverse of  $m$  modulo  $3^{2v}$ . Now, the numerator in the first exponential in the last line of (4.22) takes the form  $mx^3 - 3nx^2 \equiv -4 \cdot \overline{m}^2 n^3 \bmod 3^{2v}$ . Moreover the numerator in the second exponential in the last line of (4.22) simplifies into  $nh^2 \cdot 3^{2v-1}$ . It follows that

$$E(3^u) = 3^v e \left( \frac{-4 \cdot \overline{m}^2 n^3}{3^{2v}} \right) \sum_{h=1}^3 e \left( \frac{nh^2}{3} \right).$$

Evaluating the quadratic Gauss sum on the right-hand side, we obtain

$$E(3^u) = 3^{v+1/2} e \left( \frac{-4 \cdot \overline{m}^2 n^3}{3^{2v}} \right) \left( \frac{n}{3} \right) \epsilon_3 = 3^{(u+1)/2} e \left( \frac{-4 \cdot \overline{m}^2 n^3}{3^u} \right) \left( \frac{n}{3} \right) \epsilon_3. \quad (4.23)$$

Next suppose that  $u$  is odd. Note that  $v \geq 2$  since  $u \geq 5$ . Here we slightly differ from (4.16) and observe that

$$\begin{aligned} E(3^u) &= \sum_{x \bmod 3^v}^* \sum_{y \bmod 3^{v+1}} e \left( \frac{m(x + y \cdot 3^v)^3 - 3n(x + y \cdot 3^v)^2}{3^{2v+1}} \right) \\ &= \sum_{x \bmod 3^v}^* e \left( \frac{mx^3 - 3nx^2}{3^{2v+1}} \right) \sum_{y \bmod 3^{v+1}} e \left( \frac{mx^2 - 2nx}{3^v} \cdot y \right) \\ &= 3^{v+1} \sum_{\substack{x \bmod 3^v \\ mx \equiv 2n \bmod 3^v}}^* e \left( \frac{mx^3 - 3nx^2}{3^{2v+1}} \right). \end{aligned} \quad (4.24)$$

The only solution of the congruence in the last line can be lifted uniquely to the solution

$$x \equiv \overline{m} \cdot 2n \bmod 3^{2v+1},$$

where  $\overline{m}$  is the multiplicative inverse of  $m$  modulo  $3^{2v+1}$ . Now the numerator in the exponential in the last line of (4.24) takes the form  $mx^3 - 3nx^2 \equiv -4 \cdot \overline{m}^2 n^3 \pmod{3^{2v+1}}$ . Hence

$$E(3^u) = 3^{v+1} e\left(\frac{-4 \cdot \overline{m}^2 n^3}{3^{2v+1}}\right) = 3^{(u+1)/2} e\left(\frac{-4 \cdot \overline{m}^2 n^3}{3^u}\right). \quad (4.25)$$

We combine (4.23) and (4.25) into a single equation, namely

$$E(3^u) = 3^{(u+1)/2} e\left(\frac{-4 \cdot \overline{m}^2 n^3}{3^u}\right) \left(\frac{n}{3^{u-1}}\right) \epsilon_{3^{u-1}}. \quad (4.26)$$

Combining (4.14), (4.21) and (4.26), we obtain (4.13) for  $p = 3$ .

*Case III:  $p = 2$ .* — We note that  $f_2 = 1$  by (4.6) and use the notations

$$m := c_1, \quad n := 3^{f_3} d_1. \quad (4.27)$$

Suppose first that  $u$  is even. Note that  $v \geq 3$  since  $u \geq 6$ . From (4.15), it follows that

$$\begin{aligned} E(2^u) &= 2^v \sum_{\substack{x \pmod{2^v} \\ 3mx \equiv n \pmod{2^{v-1}}}}^* e\left(\frac{2mx^3 - nx^2}{2^{2v}}\right) \\ &= 2^v \sum_{h=1}^2 e\left(\frac{2m(x+h \cdot 2^{v-1})^3 - n(x+h \cdot 2^{v-1})^2}{2^{2v}}\right) \\ &= 2^v e\left(\frac{2mx^3 - nx^2}{2^{2v}}\right) \\ &\quad \times \sum_{h=1}^2 e\left(\frac{3mx^2 h \cdot 2^v + 6mxh^2 \cdot 2^{2v-2} - nxh \cdot 2^v - nh^2 \cdot 2^{2v-2}}{2^{2v}}\right), \end{aligned} \quad (4.28)$$

where in the last two lines,  $x$  satisfies the congruence  $3mx \equiv n \pmod{2^{v-1}}$ . The only solution of this congruence can be lifted uniquely to the solution

$$x \equiv \overline{3m} \cdot n \pmod{2^{2v}},$$

where  $\overline{m}$  is the multiplicative inverse of  $m$  modulo  $2^{2v}$ . Now the numerator in the first exponential in the last line of (4.28) takes the form  $2mx^3 - nx^2 \equiv -\overline{27} \cdot \overline{m}^2 n^3 \pmod{2^{2v}}$ . Moreover the numerator in the second exponential in the last line of (4.28) simplifies into  $nh^2 \cdot 2^{2v-2}$ . It follows that

$$\begin{aligned} E(2^u) &= 2^v e\left(\frac{-\overline{27} \cdot \overline{m}^2 n^3}{2^{2v}}\right) \sum_{h=1}^2 e\left(\frac{nh^2}{4}\right) \\ &= 2^v e\left(\frac{-\overline{27} \cdot \overline{m}^2 n^3}{2^{2v}}\right) \left(1 + e\left(\frac{n}{4}\right)\right) \\ &= 2^{(u+1)/2} e\left(\frac{-\overline{27} \cdot \overline{m}^2 n^3}{2^u}\right) \cdot \frac{(1 + e(n/4))}{\sqrt{2}}. \end{aligned} \quad (4.29)$$

Next suppose that  $u$  is odd. Note that  $v \geq 3$  since  $u \geq 7$ . From (4.16), it follows that

$$\begin{aligned}
E(2^u) &= 2^v \sum_{\substack{x \bmod 2^{v+1} \\ 3mx \equiv n \bmod 2^{v-1}}}^* e\left(\frac{2mx^3 - nx^2}{2^{2v+1}}\right) \\
&= 2^v \sum_{h=1}^4 e\left(\frac{2m(x+h \cdot 2^{v-1})^3 - n(x+h \cdot 2^{v-1})^2}{2^{2v+1}}\right) \\
&= 2^v \cdot e\left(\frac{2mx^3 - nx^2}{2^{2v+1}}\right) \\
&\quad \times \sum_{h=1}^4 e\left(\frac{3mx^2h \cdot 2^v + 6mxh^2 \cdot 2^{2v-2} - nxh \cdot 2^v - nh^2 \cdot 2^{2v-2}}{2^{2v+1}}\right),
\end{aligned} \tag{4.30}$$

where in the last two lines,  $x$  satisfies the congruence  $3mx \equiv n \bmod 2^{v-1}$ . The only solution of the congruence in the last line can be lifted uniquely to the solution

$$x \equiv \overline{3m} \cdot n \bmod 2^{2v+1},$$

where  $\overline{m}$  is the multiplicative inverse of  $m$  modulo  $2^{2v+1}$ . Now the numerator in the first exponential in the last line of (4.30) takes the form  $2mx^3 - nx^2 \equiv -\overline{27} \cdot \overline{m}^2 n^3 \bmod 2^{2v+1}$ , and the numerator in the second exponential simplifies into  $nh^2 \cdot 2^{2v-2}$ . Hence

$$E(2^u) = 2^v e\left(\frac{-\overline{27} \cdot \overline{m}^2 n^3}{2^{2v+1}}\right) \sum_{h=1}^4 e\left(\frac{nh^2}{8}\right).$$

Recall that we suppose  $(n, 2) = 1$ . In this case it is easily seen that

$$\sum_{h=1}^4 e\left(\frac{nh^2}{8}\right) = 2e\left(\frac{n}{8}\right).$$

Hence

$$E(2^u) = 2^{v+1} \cdot e\left(\frac{-\overline{27} \cdot \overline{m}^2 n^3}{2^{2v+1}}\right) \cdot e\left(\frac{n}{8}\right) = 2^{(u+1)/2} \cdot e\left(\frac{-\overline{27} \cdot \overline{m}^2 n^3}{2^u}\right) \cdot e\left(\frac{n}{8}\right). \tag{4.31}$$

We combine (4.29) and (4.31) into a single equation, namely

$$E(2^u) = 2^{(u+1)/2} \cdot e\left(\frac{-\overline{27} \cdot \overline{m}^2 n^3}{2^u}\right) \cdot \left\{ \frac{n}{2^u} \right\}, \tag{4.32}$$

where the residue symbol  $\left\{ \frac{\cdot}{2^u} \right\}$  is defined as in (4.12). Combining (4.14), (4.27) and (4.32), we obtain (4.13) for  $p = 2$ .  $\square$

**4.2. Composite moduli.** — Now we look at composite moduli  $q$ , beginning with the following general estimate.

**Lemma 4.6.** — *Let  $c, d, q \in \mathbb{Z}$  such that  $q > 0$  and let  $f_2, f_3 \in \{0, 1\}$ . Then we have*

$$|E_{f_2, f_3}(c, d; q)| \leq C(c, d, q)^{1/2} q^{1/2} 2^{\omega(q)},$$

where  $C = 10000$ .

*Proof.* — Combining all of our results from the previous section we deduce that the estimate in the lemma holds with  $C = 1$  if  $q$  is a power of a prime  $p > 3$  and with  $C = 100$  if  $q$  is a power of 2 or 3. This together with Lemma 4.2 implies the statement of the lemma for general  $q$ .  $\square$

The following lemma contains a precise evaluation of  $E_{f_2, f_3}(c, d; q)$  for power-full moduli  $q$ . Throughout the sequel, we denote by  $\text{rad}(n)$  the largest square-free divisor of  $n \in \mathbb{N}$ .

**Lemma 4.7.** — Let  $c, d, q \in \mathbb{Z}$  such that  $q > 0$ , let  $f_1, f_2 \in \{0, 1\}$  and assume that (4.6) holds. Suppose that  $q = 2^{t_2} 3^{t_3} q_1$ , where  $(q_1, 6) = 1$ . Set  $e_2 := \text{sign}(t_2)$  and  $e_3 := \text{sign}(t_3)$ . Then we have the following.

(i) If  $2^{6e_2} 3^{5e_3} \text{rad}(q_1)^2$  divides  $q/(c, q)$  or  $q/(d, q)$ , then

$$E_{f_2, f_3}(c, d; q) \neq 0 \Rightarrow (c, q) = (d, q). \quad (4.33)$$

In this case, set

$$\begin{aligned} q' &= (c, q) = (d, q), \quad \tilde{q} = \frac{q}{q'}, \quad c' = \frac{c}{q'}, \quad d' = \frac{d}{q'}, \\ q'_1 &= (c, q_1) = (d, q_1), \quad \tilde{q}_1 = \frac{q_1}{q'_1}. \end{aligned} \quad (4.34)$$

Then we have

$$E_{f_2, f_3}(c, d; q) = \frac{q}{\sqrt{\tilde{q}}} \cdot e \left( \frac{-A \cdot \overline{c'}^2 d'^3}{\tilde{q}} \right) \left( \frac{d'}{\tilde{q}_1} \right) \cdot P, \quad (4.35)$$

where  $|P| \leq 2\sqrt{3}$  depends at most on  $\tilde{q}$ ,  $f_3$  and the residue class of  $d'$  modulo  $8^{e_2} 3^{e_3}$ , and moreover,

$$A \equiv 4^{1-f_2} \cdot 27^{f_3-1} \pmod{\tilde{q}}. \quad (4.36)$$

(ii) If  $E_{f_2, f_3}(c, d; q) \neq 0$ , then the following equivalences hold:

$$\begin{aligned} \mu^2 \left( \frac{q_1}{(c, q_1)} \right) = 1 &\iff \mu^2 \left( \frac{q_1}{(d, q_1)} \right) = 1, \\ \frac{2^{t_2}}{(c, 2^{t_2})} \mid 2^5 &\iff \frac{2^{t_2}}{(d, 2^{t_2})} \mid 2^5, \\ \frac{3^{t_3}}{(c, 3^{t_3})} \mid 3^4 &\iff \frac{3^{t_3}}{(d, 3^{t_3})} \mid 3^4. \end{aligned}$$

*Proof.* — Part (ii) and the implication (4.33) in part (i) are straightforward consequences of Lemmas 4.2 and 4.4. It remains to establish (4.35) under the conditions of part (i). Throughout the following, we set

$$\begin{aligned} 2^{s_2} &= (c, 2^{t_2}) = (d, 2^{t_2}), \quad u_2 = t_2 - s_2 \geq 6e_2, \\ 3^{s_3} &= (c, 3^{t_3}) = (d, 3^{t_3}), \quad u_3 = t_3 - s_3 \geq 5e_3. \end{aligned} \quad (4.37)$$

First we assume that  $t_2 = 0$ , so that  $q$  is odd. In this case we shall prove that

$$E_{f_2, f_3}(c, d; q) = \frac{q}{\sqrt{\tilde{q}/3^{e_3}}} \cdot e \left( \frac{-A \cdot \overline{c'}^2 d'^3}{\tilde{q}} \right) \left( \frac{3^{f_3-e_3} d'}{w \tilde{q}_1} \right) \epsilon_{w \tilde{q}_1}, \quad (4.38)$$

where  $w = 3^{u_3-e_3}$ .

We first treat the case  $t_2 = 0 = t_3$ , which is equivalent to  $\tilde{q}_1 = \tilde{q}$ . In this case (4.38) takes the form

$$E_{f_2, f_3}(c, d; q) = \frac{q}{\sqrt{\tilde{q}}} \cdot e \left( \frac{-A \cdot \overline{c'}^2 d'^3}{\tilde{q}} \right) \left( \frac{3^{f_3} d'}{\tilde{q}} \right) \epsilon_{\tilde{q}}. \quad (4.39)$$

This holds trivially if  $q = 1$ . For  $q > 1$  and  $(q, 6) = 1$ , we prove (4.39) by induction over the number of prime divisors of  $q$ . If  $q = p^t$  for some prime  $p \neq 2, 3$  and  $p^2$  divides  $q/(c, q)$ , then (4.39) coincides with (4.13). Now assume that  $g$  is the number of prime divisors of  $q$  and that (4.39) has been established for all moduli  $r$  with  $g-1$  prime divisors such that  $(r, 6) = 1$  and  $\text{rad}(r)^2$  divides

$r/(r, d)$ . Suppose that  $q = r_1 r_2$ , where  $(r_1, r_2) = 1$  and  $r_2 = p^t$  is a prime power. Let  $\overline{r_1}, \overline{r_2}$  be such that  $r_1 \overline{r_1} + r_2 \overline{r_2} = 1$ . Then using (4.7), the induction hypothesis and Lemma 4.5, we have

$$\begin{aligned} E_{f_2, f_3}(c, d; q) &= E_{f_2, f_3}(\overline{c r_2}, \overline{d r_2}; r_1) E_{f_2, f_3}(\overline{c r_1}, \overline{d r_1}; r_2) \\ &= \left( \frac{r_1}{\sqrt{\tilde{r}_1}} \cdot e \left( \frac{-A \cdot \overline{r_2} \cdot \overline{c_1^2 d_1^3}}{\tilde{r}_1} \right) \left( \frac{3^{f_3} \overline{r_2} d_1}{\tilde{r}_1} \right) \epsilon_{\tilde{r}_1} \right) \\ &\quad \times \left( \frac{r_2}{\sqrt{\tilde{r}_2}} \cdot e \left( \frac{-A \cdot \overline{r_1} \cdot \overline{c_2^2 d_2^3}}{\tilde{r}_2} \right) \left( \frac{3^{f_3} \overline{r_1} d_2}{\tilde{r}_2} \right) \epsilon_{\tilde{r}_2} \right), \end{aligned} \quad (4.40)$$

where for  $i = 1, 2$  we set

$$r'_i = (c, r_i) = (d, r_i), \quad \tilde{r}_i = \frac{r_i}{r'_i}, \quad c_i = \frac{c}{r'_i}, \quad d_i = \frac{d}{r'_i}.$$

We note that

$$\begin{aligned} c_1 &= \frac{c}{r'_1} = \frac{c r'_2}{q'} = c' r'_2, & c_2 &= \frac{c}{r'_2} = \frac{c r'_1}{q'} = c' r'_1, \\ d_1 &= \frac{d}{r'_1} = \frac{d r'_2}{q'} = d' r'_2, & d_2 &= \frac{d}{r'_2} = \frac{d r'_1}{q'} = d' r'_1. \end{aligned}$$

Combining the exponential terms we obtain

$$\begin{aligned} &e \left( \frac{-A \cdot \overline{r_2} \cdot \overline{c_1^2 d_1^3}}{\tilde{r}_1} \right) \cdot e \left( \frac{-A \cdot \overline{r_1} \cdot \overline{c_2^2 d_2^3}}{\tilde{r}_2} \right) \\ &= e \left( \frac{-A \cdot \overline{r_1} \cdot \overline{c' r'_1^2} (d' r'_1)^3}{\tilde{r}_2} \right) \cdot e \left( \frac{-A \cdot \overline{r_2} \cdot \overline{c' r'_2^2} (d' r'_2)^3}{\tilde{r}_1} \right) \\ &= e \left( \frac{-A \cdot \overline{c'^2 d'^3} (\overline{r_2} r'_2 \tilde{r}_2 + \overline{r_1} r'_1 \tilde{r}_1)}{\tilde{r}_1 \tilde{r}_2} \right) \\ &= e \left( \frac{-A \cdot \overline{c'^2 d'^3}}{\tilde{q}} \right), \end{aligned}$$

where  $\overline{c'} \equiv 1 \pmod{\tilde{q}}$ . Furthermore, by the multiplicativity of the Jacobi symbol, we have

$$\begin{aligned} \left( \frac{3^{f_3} \overline{r_2} d_1}{\tilde{r}_1} \right) \left( \frac{3^{f_3} \overline{r_1} d_2}{\tilde{r}_2} \right) &= \left( \frac{3^{f_3} \overline{r_2} d' r'_2}{\tilde{r}_1} \right) \left( \frac{3^{f_3} \overline{r_1} d' r'_1}{\tilde{r}_2} \right) \\ &= \left( \frac{3^{f_3} \overline{r_2} d' r'_2 / \tilde{r}_2}{\tilde{r}_1} \right) \left( \frac{3^{f_3} \overline{r_1} d' r'_1 / \tilde{r}_1}{\tilde{r}_2} \right) \\ &= \left( \frac{3^{f_3} d'}{\tilde{r}_1} \right) \left( \frac{3^{f_3} d'}{\tilde{r}_2} \right) \left( \frac{\overline{r_2} r'_2 / \tilde{r}_2}{\tilde{r}_1} \right) \left( \frac{\overline{r_1} r'_1 / \tilde{r}_1}{\tilde{r}_2} \right) \\ &= \left( \frac{3^{f_3} d'}{\tilde{q}} \right) \left( \frac{\tilde{r}_1}{\tilde{r}_2} \right) \left( \frac{\tilde{r}_2}{\tilde{r}_1} \right). \end{aligned}$$

Moreover, by quadratic reciprocity,

$$\left( \frac{\tilde{r}_1}{\tilde{r}_2} \right) \left( \frac{\tilde{r}_2}{\tilde{r}_1} \right) \epsilon_{\tilde{r}_1} \epsilon_{\tilde{r}_2} = \epsilon_{\tilde{r}_1 \tilde{r}_2} = \epsilon_{\tilde{q}}.$$

Combining these with (4.40) we obtain (4.39).

Next, we turn to the case when  $t_2 = 0$  and  $t_3 > 0$ . Note that  $f_3 = e_3 = 1$ . Using (4.7), we have

$$E_{f_2, f_3}(c, d; q) = E(2^{f_2} c, 3d; q) = E(2^{f_2} 3^{-t_3} c, 3^{1-t_3} d; q_1) E(2^{f_2} \overline{c q_1}, 3 \overline{d q_1}; 3^{t_3}).$$



Applying (4.39) and Lemma 4.5 for  $p = 3$  to the right-hand side, we obtain

$$\begin{aligned} E(2^{f_2}c, 3^{f_3}d; q) &= \frac{q_1}{\sqrt{\tilde{q}_1}} \cdot e\left(\frac{-A \cdot 3^{-t_3} \cdot \overline{c_1}^2 d_1^3}{\tilde{q}_1}\right) \left(\frac{3^{1-t_3} d_1}{\tilde{q}_1}\right) \epsilon_{\tilde{q}_1} \\ &\quad \times 3^{t_3-(u_3-1)/2} \cdot e\left(\frac{-A \cdot \overline{q_1} \cdot \overline{c_2}^2 d_2^3}{3^{u_3}}\right) \left(\frac{\overline{q_1} d_2}{3^{u_3-1}}\right) \epsilon_w, \end{aligned}$$

where  $w = 3^{u_3-e_3}$  and

$$c_1 = \frac{c}{q'_1}, \quad d_1 = \frac{d}{q'_1}, \quad c_2 = \frac{c}{3^{s_3}}, \quad d_2 = \frac{d}{3^{s_3}}.$$

Recall the definitions in (4.34) and (4.37). Proceeding along the same lines as in the case  $(q, 6) = 1$ , we deduce that

$$E_{f_2, f_3}(c, d; q) = \frac{q}{\sqrt{\tilde{q}/3}} \cdot e\left(\frac{-A \cdot \overline{c'}^2 d'^3}{\tilde{q}}\right) \left(\frac{d'}{w\tilde{q}_1}\right) \epsilon_{w\tilde{q}_1},$$

which establishes (4.38) for the case  $t_2 = 0$  and  $t_3 > 0$  and so completes its proof.

From (4.38), it follows that if  $t_2 = 0$  then (4.35) holds with

$$P = \sqrt{3^{e_3}} \left(\frac{3^{f_3-e_3}}{w\tilde{q}_1}\right) \left(\frac{d'}{w}\right) \epsilon_{w\tilde{q}_1}.$$

Thus  $|P| \leq \sqrt{3}$  and one sees that  $P$  depends at most on  $\tilde{q}, f_3$  and the residue class of  $d'$  modulo 3.

It remains to consider the case when  $t_2 > 0$ , for which  $f_2 = e_2 = 1$ . Using (4.7) we have

$$E_{f_2, f_3}(c, d; q) = E(2c, 3^{f_3}d; q) = E(2^{1-t_2}c, 2^{-t_2}3^{f_3}d; 3^{t_3}q_1) E(2 \cdot 3^{-t_3} \overline{q_1} c, 3^{f_3-t_3} \overline{q_1} d; 2^{t_2}).$$

Applying Lemma 4.5 for  $p = 2$  and (4.38) to the right-hand side, we obtain

$$\begin{aligned} E_{f_2, f_3}(c, d; q) &= \frac{3^{t_3} q_1}{\sqrt{3^{u_3-e_3} \tilde{q}_1}} \cdot e\left(\frac{-A \cdot 2^{-t_2} \cdot \overline{c_1}^2 d_1^3}{3^{u_3} \tilde{q}_1}\right) \left(\frac{2^{-t_2} 3^{f_3-e_3} d_1}{w\tilde{q}_1}\right) \epsilon_{w\tilde{q}_1} \\ &\quad \times 2^{t_2-(u_2-1)/2} \cdot e\left(\frac{-A \cdot 3^{-t_3} \cdot \overline{q_1} \cdot \overline{c_2}^2 d_2^3}{2^{u_2}}\right) \left\{ \frac{3^{f_3-t_3} \overline{q_1} d_2}{2^{u_2}} \right\}, \end{aligned}$$

where

$$c_1 = \frac{c}{3^{s_3} q'_1}, \quad d_1 = \frac{d}{3^{s_3} q'_1}, \quad c_2 = \frac{c}{2^{s_2}}, \quad d_2 = \frac{d}{2^{s_2}}.$$

Proceeding along the same lines as in the case  $(q, 6) = 1$ , we deduce that

$$E_{f_2, f_3}(c, c; q) = \frac{q}{\sqrt{\tilde{q}/(2 \cdot 3^{e_3})}} \cdot e\left(\frac{-A \cdot \overline{c'}^2 d'^3}{\tilde{q}}\right) \left(\frac{2^{u_2} 3^{f_3-e_3} d'}{w\tilde{q}_1}\right) \epsilon_{w\tilde{q}_1} \cdot \left\{ \frac{3^{f_3-u_3} \overline{q_1} d'}{2^{u_2}} \right\}.$$

It follows that (4.35) holds with

$$P = \sqrt{2 \cdot 3^{e_3}} \cdot \left(\frac{2^{u_2} 3^{f_3-e_3}}{w\tilde{q}_1}\right) \left(\frac{d'}{w}\right) \epsilon_{w\tilde{q}_1} \cdot \left\{ \frac{3^{f_3-u_3} \overline{q_1} d'}{2^{u_2}} \right\}$$

in this case. Recalling the definition of the residue symbol (4.12) one easily deduces that  $|P| \leq 2\sqrt{3}$  and  $P$  depends only on  $\tilde{q}, f_3$  and the residue of  $d'$  modulo  $8 \cdot 3^{e_3}$ .  $\square$

Since it will turn out to be very convenient to have the condition

$$2^{6e_2} 3^{5e_3} \text{rad}(q_1)^2 \mid \frac{q}{(c, q)}$$

in part (i) of Lemma 4.7 replaced by the weaker condition

$$\text{rad}(q_1)^2 \mid \frac{q}{(c, q)},$$

we provide a result based on the previous one. Let  $v_p(n)$  the  $p$ -adic valuation of a natural number  $n$  and recall the definition (4.8) of  $r(p)$ . Then we have the following result.

**Lemma 4.8.** — *Let  $c, d, q \in \mathbb{Z}$  such that  $q > 0$ , let  $f_2, f_3 \in \{0, 1\}$  and assume that (4.6) holds, with  $E_{f_2, f_3}(c, d; q) \neq 0$ . Suppose that  $q = 2^{l_2} 3^{l_3} q_1$ , where  $(q_1, 6) = 1$ . Set*

$$q' := (c, q), \quad \tilde{q} := \frac{q}{q'}, \quad q'_1 := (c, q_1), \quad \tilde{q}_1 := \frac{q_1}{q'_1}, \quad c' := \frac{c}{(c, q)} = \frac{c}{q'}, \quad d' := \frac{d}{(d, q)}.$$

Suppose that  $\text{rad}(q_1)^2 \mid \tilde{q}$ . Let

$$q_0 := \prod_{\substack{p \\ v_p(\tilde{q}) \geq r(p)}} p^{v_p(q)}, \quad q'_0 := (c, q_0), \quad \tilde{q}_0 := \frac{q_0}{q'_0},$$

and

$$j_p := v_p((d, q)) - v_p(q'). \quad (4.41)$$

Then we have

$$\frac{\tilde{q}}{\tilde{q}_0} = 2^{l_2} 3^{l_3} \quad \text{for some } l_2, l_3 \in \mathbb{Z} \text{ with } 0 \leq l_2 \leq 5 \text{ and } 0 \leq l_3 \leq 4, \quad (4.42)$$

$$|j_2| \leq 5, \quad |j_3| \leq 4, \quad j_p = 0 \text{ if } p > 3. \quad (4.43)$$

Furthermore

$$E_{f_2, f_3}(c, d; q) = \frac{q}{\sqrt{\tilde{q}}} \cdot e \left( \frac{-2^{g_2} 3^{g_3} \cdot \overline{c'}^2 d'^3}{\tilde{q}_0} \right) \left( \frac{d'}{\tilde{q}_1} \right) \cdot Q, \quad (4.44)$$

where  $g_2, g_3 \in \mathbb{Z}$  depend at most on  $q, q', f_2, f_3$  and satisfy

$$|g_2| \leq 20, \quad |g_3| \leq 17, \quad g_p = 0 \text{ if } p \mid \tilde{q}_0 \text{ for } p = 2, 3,$$

and  $Q$  depends at most on  $q, q', f_2, f_3, j_2, j_3$  and the residue classes of  $c'$  and  $d'$  modulo  $2^5 3^4$  and satisfies  $Q = O(1)$ , with an absolute implied constant.

*Proof.* — The assertion in (4.42) follows from the definitions of  $\tilde{q}$  and  $\tilde{q}_0$  and the hypothesis that  $\text{rad}(q_1)^2 \mid \tilde{q}$ . The relations in (4.43) are consequences of this assumption and part (ii) and (4.33) in part (i) of Lemma 4.7. It remains to show (4.44) with  $g_2, g_3$  and  $Q$  having the claimed properties.

By the definition of  $q_0$  and the assumption  $\text{rad}(q_1)^2 \mid \tilde{q}$ , we have

$$q = 2^{h_2} 3^{h_3} q_0, \quad (4.45)$$

with

$$h_p = \begin{cases} t_p & \text{if } p \nmid q_0, \\ 0 & \text{if } p \mid q_0. \end{cases}$$

In particular  $(2^{h_2} 3^{h_3}, q_0) = 1$ . From (4.7) it therefore follows that

$$\begin{aligned} E_{f_2, f_3}(c, d; q) &= E_{f_2, f_3}(2^{-h_2} 3^{-h_3} c, 2^{-h_2} 3^{-h_3} d; q_0) E_{f_2, f_3}(\overline{q_0} c, \overline{q_0} d; 2^{h_2} 3^{h_3}) \\ &= E^{(1)} E^{(2)}, \end{aligned} \quad (4.46)$$

say. We can apply Lemma 4.7 to the first exponential sum on the right-hand side, obtaining

$$(c, q_0) = q'_0 = (d, q_0)$$

since this exponential sum is non-zero by hypothesis. Furthermore we have

$$E^{(1)} = \frac{q_0}{\sqrt{\tilde{q}_0}} \cdot e \left( \frac{-A \cdot 2^{-h_2} 3^{-h_3} \overline{c''}^2 d''^3}{\tilde{q}_0} \right) \left( \frac{2^{h_2} 3^{h_3} d''}{\tilde{q}_1} \right) \cdot P,$$

where  $A$  is defined as in (4.36),

$$c'' = \frac{c}{q'_0}, \quad d'' = \frac{d}{q'_0},$$

and  $P = O(1)$  depends only on  $\tilde{q}_0$ ,  $f_3$  and the (well-defined) residue class of  $2^{-h_2}3^{-h_3}d''$  modulo  $8^{e_2}3^{e_3}$ , with  $e_p := \text{sign}(v_p(\tilde{q}_0))$ . Moreover, we may write

$$c'' = c' \cdot \frac{q'}{q'_0} \quad \text{and} \quad d'' = d' \cdot \frac{(d, q)}{q'_0}.$$

By (4.41) and (4.43), we have

$$d'' = d' \cdot \frac{q'}{q'_0} \cdot 2^{j_2}3^{j_3}$$

with  $|j_2| \leq 5$  and  $|j_3| \leq 4$ . The construction of  $q_0$  ensures that we have  $j_p = 0$  if  $p \mid \tilde{q}_0$  for  $p = 2, 3$ . Combining this with our work so far we obtain

$$E^{(1)} = \frac{q_0}{\sqrt{\tilde{q}_0}} \cdot e \left( \frac{-A \cdot 2^{3j_2-h_2}3^{3j_3-h_3} \cdot \frac{q'}{q'_0} \cdot \overline{c'}^2 d'^3}{\tilde{q}_0} \right) \left( \frac{2^{h_2+j_2}3^{h_3+j_3} \cdot \frac{q'}{q'_0} \cdot d'}{\tilde{q}_1} \right) \cdot P.$$

In view of (4.45) we have

$$\frac{q'}{q'_0} = \frac{(c, 2^{h_2}3^{h_3}q_0)}{(c, q_0)} = (c, 2^{h_2}3^{h_3}),$$

whence  $0 \leq v_p(q'/q'_0) \leq h_p$ . It is now easy to deduce that  $v_p(q'/q'_0) = 0$  if  $p > 3$  and

$$\max\{0, h_2 - 5\} \leq v_2 \left( \frac{q'}{q'_0} \right) \leq h_2, \quad \max\{0, h_3 - 4\} \leq v_3 \left( \frac{q'}{q'_0} \right) \leq h_3. \quad (4.47)$$

It follows that

$$E^{(1)} = \frac{q_0}{\sqrt{\tilde{q}_0}} \cdot e \left( \frac{-2^{g_2}3^{g_3} \cdot \overline{c'}^2 d'^3}{\tilde{q}_0} \right) \left( \frac{d'}{\tilde{q}_1} \right) \cdot P', \quad (4.48)$$

where  $g_2$  and  $g_3$  have the properties claimed in Lemma 4.8 (where we also use that  $f_p = 1$  if  $p \mid \tilde{q}_0$  for  $p = 2, 3$ ), and  $|P'| \leq 2\sqrt{3}$  depends only on  $q$ ,  $q'$ ,  $f_3$ ,  $j_2$ ,  $j_3$  and the residue class of  $d'$  modulo  $8^{e_2}3^{e_3}$ . Here and later, note that  $q$ ,  $q'$  determine  $\tilde{q}$ ,  $q_0$ ,  $q'_0$ ,  $\tilde{q}_0$ ,  $q_1$ ,  $q'_1$ ,  $\tilde{q}_1$ ,  $h_2$  and  $h_3$ .

Now we turn to the second exponential sum on the right-hand side of (4.46). We write

$$c = q'c', \quad d = 2^{j_2}3^{j_3}q'd',$$

and hence

$$\begin{aligned} E^{(2)} &= E_{f_2, f_3}(\overline{q_0}q'c', 2^{j_2}3^{j_3}\overline{q_0}q'd'; 2^{h_2}3^{h_3}) \\ &= E_{f_2, f_3} \left( \overline{q_0} \cdot \frac{q'}{q'_0} \cdot c', \overline{q_0} \cdot 2^{j_2}3^{j_3} \cdot \frac{q'}{q'_0} \cdot d'; 2^{h_2}3^{h_3} \right), \end{aligned}$$

where  $2^{j_2}3^{j_3} \cdot q'/q'_0 \in \mathbb{N}$ . In particular  $|E^{(2)}| \leq 2^{h_2}3^{h_3}$  and  $E^{(2)}$  depends only on  $q$ ,  $q'$ ,  $f_2$ ,  $f_3$ ,  $j_2$ ,  $j_3$  and the residue classes of  $c'$ ,  $d'$  modulo  $2^5 3^4$ . To see the latter claim we note that we are only interested in  $\overline{q_0}C \cdot c'$  and  $\overline{q_0}D \cdot d'$  modulo  $2^{h_2}3^{h_3}$ , where  $C = q'/q'_0$  and  $D = 2^{j_2}3^{j_3}q'/q'_0$ . But then the claim follows from (4.43) and (4.47). Once combined with (4.48) this therefore yields (4.44) with  $g_2, g_3$  and  $Q$  having the properties recorded in the lemma.  $\square$

### 5. Weighted solutions of $ax^2 + by^3 \equiv 0 \pmod{q}$

Let  $\Gamma$  denote the Gaussian weight in (3.1). We note that the Fourier transform  $\hat{\Gamma}$  of  $\Gamma$  satisfies

$$\hat{\Gamma}(x) := \int_{-\infty}^{\infty} \Gamma(t) e(-tx) dt = \exp(-\pi x^2) = \Gamma(x)$$

for all  $x \in \mathbb{R}$ . Our starting point for the analysis of the counting functions  $M(B, \mathbf{X}, \mathbf{Y}; a, b; q)$  and  $M(\mathbf{X}, \mathbf{Y}; a, b; q)$  from §1 lies with an initial investigation of the weighted sum

$$\mathcal{S}(x_0, y_0, X, Y; a, b; q) := \sum_{\substack{x, y \in \mathbb{Z} \\ (xy, q) = 1 \\ ax^2 + by^3 \equiv 0 \pmod{q}}} \Gamma\left(\frac{x - x_0}{X}\right) \Gamma\left(\frac{y - y_0}{Y}\right), \quad (5.1)$$

for given  $x_0, y_0 \in \mathbb{R}$  and  $X, Y \geq 2$ . We will write  $\mathcal{S} = \mathcal{S}(x_0, y_0, X, Y; a, b; q)$  for short and we note that without loss of generality we may assume that  $a, b > 0$  in all that follows.

We analyse  $\mathcal{S}$  via Poisson summation, the first task being to break the sum into residue classes modulo  $q$ . Thus we may write

$$\mathcal{S} = \sum_{\substack{c, d \pmod{q} \\ ac^2 + bd^3 \equiv 0 \pmod{q}}}^* \sum_{x \equiv c \pmod{q}} \Gamma\left(\frac{x - x_0}{X}\right) \sum_{y \equiv d \pmod{q}} \Gamma\left(\frac{y - y_0}{Y}\right),$$

where the asterisk attached to the summation symbol indicates coprimality of the variables  $c, d$  with  $q$ . Applying Poisson summation after a linear change of variables to the sums over  $x$  and  $y$  on the right-hand side, we obtain

$$\begin{aligned} \sum_{x \equiv c \pmod{q}} \Gamma\left(\frac{x - x_0}{X}\right) &= \frac{X}{q} \sum_{m \in \mathbb{Z}} \Gamma\left(\frac{mX}{q}\right) e\left(\frac{(c - x_0)m}{q}\right), \\ \sum_{y \equiv d \pmod{q}} \Gamma\left(\frac{y - y_0}{Y}\right) &= \frac{Y}{q} \sum_{n \in \mathbb{Z}} \Gamma\left(\frac{nY}{q}\right) e\left(\frac{(d - y_0)n}{q}\right). \end{aligned}$$

Hence

$$\mathcal{S} = \frac{XY}{q^2} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} e\left(-\frac{mx_0 + ny_0}{q}\right) \Gamma\left(\frac{mX}{q}\right) \Gamma\left(\frac{nY}{q}\right) \mathcal{E}(m, n; q), \quad (5.2)$$

where

$$\mathcal{E}(m, n; q) := \sum_{\substack{c, d \pmod{q} \\ ac^2 + bd^3 \equiv 0 \pmod{q}}}^* e\left(\frac{cm + dn}{q}\right).$$

Inspired by the work of Pierce [22], it will be convenient to rewrite the exponential sum  $\mathcal{E}(m, n; q)$  as a sum over a single variable. Recall that  $(ab, q) = 1$ . We consider the map

$$f : \{(c, d) \in \mathbb{Z}_q^* \times \mathbb{Z}_q^* : ac^2 + bd^3 \equiv 0 \pmod{q}\} \rightarrow \mathbb{Z}_q^*,$$

defined by

$$f(c, d) := -ac\bar{d}.$$

In the above,  $\mathbb{Z}_q^*$  is the group of units of  $\mathbb{Z}_q = \mathbb{Z}/q\mathbb{Z}$ , and  $\bar{d}$  is the multiplicative inverse of  $d \in \mathbb{Z}_q^*$ . This map is bijective. Indeed, we check that

$$f^{-1}(z) := (\overline{a^2bz^3}, -\overline{ab}z^2)$$

is the inverse map. Thus

$$f^{-1} \circ f(c, d) = (-\overline{a^2b} \cdot a^3c^3\bar{d}^3, -\overline{ab} \cdot a^2c^2\bar{d}^2) = (c, d)$$

and

$$f \circ f^{-1}(z) = -a \cdot \overline{a^2 b} \cdot z^3 \cdot \overline{-ab} \cdot z^2 = z.$$

Hence we may parametrize  $c$  and  $d$  in the definition of  $\mathcal{E}(m, n; q)$  by  $c = \overline{a^2 b} z^3$  and  $d = -\overline{ab} z^2$  for  $z \in \mathbb{Z}_q^*$ . It follows that

$$\mathcal{E}(m, n; q) = \sum_{z \bmod q}^* e\left(\frac{\overline{a^2 b} m z^3 - \overline{ab} n z^2}{q}\right).$$

Making the change of variables  $z \rightarrow abz$ , the above can also be written in the form

$$\mathcal{E}(m, n; q) = \sum_{z \bmod q}^* e\left(\frac{ab^2 m z^3 - abn z^2}{q}\right) = E(ab^2 m, abn; q), \quad (5.3)$$

in the notation of (4.1).

We now split the sum  $\mathcal{S}$  in (5.2) into three. We let  $\mathcal{S}_0$  denote the contribution of  $m = 0$  and  $n = 0$ , we let  $\mathcal{S}_1$  denote the contribution of  $m \neq 0$  and  $n$  arbitrary, and we let  $\mathcal{S}_2$  denote the contribution of  $m = 0$  and  $n \neq 0$ . The treatment of  $\mathcal{S}_0$  and  $\mathcal{S}_2$  is straightforward.

The term  $\mathcal{S}_0$  will be the main term. Obviously, we have  $\mathcal{E}(0, 0; q) = \varphi(q)$  and therefore

$$\mathcal{S}_0 = \frac{\varphi(q)}{q^2} \cdot XY. \quad (5.4)$$

Since  $(ab, q) = 1$ , it follows from Lemma 4.7(ii) that  $\mathcal{E}(0, n; q) = 0$  unless  $(q/\text{rad}(q)) \mid 2^4 3^3 n$ . Setting  $2^4 3^3 n = n_1 \cdot q/\text{rad}(q)$ , we therefore get

$$\begin{aligned} \mathcal{S}_2 &= \frac{XY}{q^2} \sum_{n \neq 0} e\left(-\frac{ny_0}{q}\right) \Gamma\left(\frac{nY}{q}\right) \mathcal{E}(0, n; q) \\ &\ll \frac{XY}{q^2} \sum_{n_1 \neq 0} \left| \Gamma\left(\frac{n_1 Y}{2^4 3^3 \text{rad}(q)}\right) \right| \left| \mathcal{E}\left(0, \frac{n_1 q}{2^4 3^3 \text{rad}(q)}; q\right) \right|, \end{aligned}$$

where we use the convention that  $\mathcal{E}(0, z; q) = 0$  if  $z \notin \mathbb{Z}$ . Using Lemma 4.6, we further have

$$\left| \mathcal{E}\left(0, \frac{n_1 q}{2^4 3^3 \text{rad}(q)}; q\right) \right| \ll \frac{q}{\text{rad}(q)^{1/2}} \cdot (n_1, \text{rad}(q))^{1/2} \cdot 2^{\omega(q)}.$$

Combining these estimates with (3.18), we get

$$\mathcal{S}_2 \ll \frac{(2 + \varepsilon)^{\omega(q)} \text{rad}(q)^{1/2}}{q} \cdot X. \quad (5.5)$$

Here we have observed that  $(1 + \varepsilon)^{\omega(\text{rad}(q))} 2^{\omega(q)} \leq (2 + \varepsilon)^{\omega(q)}$ , where henceforth we adhere to the convention that  $\varepsilon$  may take different values from appearance to appearance.

We now turn to the estimation

$$\mathcal{S}_1 = \frac{XY}{q^2} \sum_{m \neq 0} \sum_n e\left(-\frac{mx_0 + ny_0}{q}\right) \Gamma\left(\frac{mX}{q}\right) \Gamma\left(\frac{nY}{q}\right) \mathcal{E}(m, n; q). \quad (5.6)$$

We always keep in mind that by (5.3) and the fact that  $(ab, q) = 1$ , statements about  $E(c, d; q)$  translate into corresponding statements about  $\mathcal{E}(m, n; q)$ . Using Lemma 4.3 we deduce that

$$\mathcal{S}_1 = \frac{XY}{q^2} \sum_{m \neq 0} \sum_n e\left(-\frac{2^{f_2} m x_0 + 3^{f_3} n y_0}{q}\right) \Gamma\left(\frac{2^{f_2} m X}{q}\right) \Gamma\left(\frac{3^{f_3} n Y}{q}\right) \mathcal{E}_{f_2, f_3}(m, n; q),$$

where  $f_i := w_i(q)$  for  $i = 2, 3$ , defined as in (4.4) and (4.5), and

$$\mathcal{E}_{f_2, f_3}(m, n; q) := \mathcal{E}(2^{f_2} m, 3^{f_3} n; q).$$

We split  $\mathcal{S}_1$  according to the greatest common divisor of  $m$  and  $q$ . Thus we write

$$\mathcal{S}_1 = \frac{XY}{q^2} \sum_{d|q} \sum_{\substack{m \neq 0 \\ (m,q)=d}} \sum_n e\left(-\frac{2^{f_2}mx_0 + 3^{f_3}ny_0}{q}\right) \Gamma\left(\frac{2^{f_2}mX}{q}\right) \Gamma\left(\frac{3^{f_3}nY}{q}\right) \mathcal{E}_{f_2,f_3}(m,n;q).$$

Throughout the sequel, we denote by  $\hat{k}$  the largest divisor coprime to 6 of  $k \in \mathbb{N}$ . Now, we uniquely factorise  $q$  and  $d$  in the form

$$q = r_1 r_2, \quad d = d_1 d_2 \text{ where } d_1 \mid r_1, d_2 \mid r_2, \\ (6r_1, r_2) = 1 = (d_1, d_2), \quad \text{rad}(\hat{r}_1)^2 \mid (r_1/d_1), \quad \mu^2(r_2/d_2) = 1.$$

The existence and uniqueness of this factorisation is obvious in the case when  $q$  is a prime power and then follows for the general case by multiplicativity. We may now write

$$\begin{aligned} \mathcal{S}_1 = \frac{XY}{q^2} \sum_{\substack{(6r_1, r_2)=1 \\ r_1 r_2 = q}} \sum_{\substack{d_1 \mid r_1, d_2 \mid r_2 \\ \text{rad}(\hat{r}_1)^2 \mid e_1 \\ \mu^2(e_2)=1}} \sum_{\substack{m \neq 0 \\ (m, r_1)=d_1 \\ (m, r_2)=d_2}} \sum_n e\left(-\frac{2^{f_2}mx_0 + 3^{f_3}ny_0}{q}\right) \\ \times \Gamma\left(\frac{2^{f_2}mX}{q}\right) \Gamma\left(\frac{3^{f_3}nY}{q}\right) \mathcal{E}_{f_2,f_3}(m,n;q), \end{aligned} \quad (5.7)$$

where

$$e_i := \frac{r_i}{d_i} \quad \text{for } i = 1, 2. \quad (5.8)$$

Using (4.7) we factorise the exponential sum as

$$\mathcal{E}_{f_2,f_3}(m,n;q) = \mathcal{E}_{f_2,f_3}(m\overline{r_2}, n\overline{r_2}; r_1) \mathcal{E}_{f_2,f_3}(m\overline{r_1}, n\overline{r_1}; r_2). \quad (5.9)$$

We shall use Lemma 4.8 to evaluate the first exponential sum on the right-hand side. To this end we recall the definition of  $r(p)$  from (4.8). Then we set

$$r_0 := \prod_{\substack{p \\ v_p(e_1) \geq r(p)}} p^{v_p(r_1)}, \quad d_0 := (m, r_0), \quad e_0 := \frac{r_0}{d_0}, \quad (5.10)$$

and

$$d'_1 := (n, r_1), \quad e'_1 := \frac{r_1}{d'_1}, \quad m_1 := \frac{m}{d'_1}, \quad n_1 := \frac{n}{d'_1}. \quad (5.11)$$

Now, from (5.3) and Lemma 4.8, we deduce that if  $\mathcal{E}_{f_2,f_3}(m\overline{r_2}, n\overline{r_2}; r_1) \neq 0$  then

$$d'_1 = 2^{j_2} 3^{j_3} d_1 \in \mathbb{N} \text{ for some } j_2, j_3 \in \mathbb{Z} \text{ with } |j_2| \leq 5 \text{ and } |j_3| \leq 4, \quad (5.12)$$

and, if this is the case, then

$$\mathcal{E}_{f_2,f_3}(m\overline{r_2}, n\overline{r_2}; r_1) = \sqrt{r_1 d_1} \cdot e\left(\frac{-2^{g_2} 3^{g_3} \cdot a \cdot \overline{br_2} \cdot \overline{m_1}^2 n_1^3}{e_0}\right) \left(\frac{n_1}{\hat{e}_1}\right) \cdot Q, \quad (5.13)$$

where  $g_2, g_3 \in \mathbb{Z}$  depend at most on  $d_1, d'_1, r_1, f_2, f_3$  and satisfy

$$|g_2| \leq 20, \quad |g_3| \leq 17, \quad g_p = 0 \text{ if } p \mid e_0 \text{ for } p = 2, 3,$$

and  $Q$  depends at most on  $a, b, d_1, d'_1, r_1, r_2, f_2, f_3$  and the residue classes of  $m_1$  and  $n_1$  modulo  $2^5 3^4$  and satisfies the bound  $Q = O(1)$ . Moreover, from (4.42), it follows that

$$e_1 = 2^{l_2} 3^{l_3} e_0 \text{ for some } l_2, l_3 \in \mathbb{Z} \text{ with } 0 \leq l_2 \leq 5 \text{ and } 0 \leq l_3 \leq 4. \quad (5.14)$$

Now we turn to the second exponential sum on the right-hand side of (5.9). Since  $r_2/d_2 = e_2$  is square-free and  $d_2 \mid m$ , we have that  $(r_2/\text{rad}(r_2)) \mid m$ . Moreover, from Lemma 4.7(ii) and the fact that  $(r_2, 6) = 1$ , we deduce

$$\mathcal{E}_{f_2,f_3}(m\overline{r_1}, n\overline{r_1}; r_2) \neq 0 \Rightarrow r_2/(n, r_2) \text{ is square-free} \Rightarrow (r_2/\text{rad}(r_2)) \mid n.$$

We set

$$d'_2 := \frac{r_2}{\text{rad}(r_2)} \quad (5.15)$$

and note that  $d'_2 \mid d_2$  since  $e_2 = r_2/d_2$  is supposed to be square-free. We further set

$$d''_2 := \frac{d_2}{d'_2} = \frac{d_2 \text{rad}(r_2)}{r_2} = \frac{\text{rad}(r_2)}{e_2}. \quad (5.16)$$

In particular  $(d''_2, e_2) = 1$  since  $\text{rad}(r_2)$  is square-free. Now it follows that

$$\mathcal{E}_{f_2, f_3}(m\overline{r_1}, n\overline{r_1}; r_2) = d'_2 \mathcal{E}_{f_2, f_3}(m_2\overline{r_1}, n_2\overline{r_1}; \text{rad}(r_2)), \quad (5.17)$$

where

$$m_2 := \frac{m}{d'_2}, \quad n_2 := \frac{n}{d'_2}. \quad (5.18)$$

We also note that

$$d_2^* := \frac{r_2 d_2^2}{(d'_2)^3} = d_2'^2 \text{rad}(r_2) = d_2'^3 e_2 \in \mathbb{N}, \quad (5.19)$$

which we will need in the following.

From (5.7) and the above considerations, we conclude that  $m = d_1 d_2 u$  for some nonzero integer  $u$  with  $(u, e_1 e_2) = 1$ , and  $n = d'_1 d'_2 v$  for some integer  $v$  with  $(v, e'_1) = 1$ . Note that if  $n = 0$  then  $e'_1 = 1$  and so  $(0, e'_1) = 1$ . By (5.11), (5.13), (5.15), (5.19) and the properties of the Jacobi symbol, we have

$$\begin{aligned} \mathcal{E}_{f_2, f_3}(m\overline{r_2}, n\overline{r_2}; r_1) &= \sqrt{r_1 d_1} \cdot e \left( \frac{-2^{g_2} 3^{g_3} a \cdot \overline{b r_2} \cdot \overline{d_2 u}^2 (d'_2 v)^3}{e_0} \right) \left( \frac{d'_2 v}{\hat{e}_1} \right) \cdot Q \\ &= \sqrt{r_1 d_1} \cdot e \left( \frac{-2^{g_2} 3^{g_3} \cdot a \cdot \overline{b d_2^*} \cdot \overline{u}^2 v^3}{e_0} \right) \left( \frac{v}{e} \right) \cdot Q', \end{aligned}$$

where  $e$  is the square-free kernel of  $\hat{e}_1$ , the unique square-free number for which  $\hat{e}_1/e$  is a perfect square, and

$$Q' := \left( \frac{d'_2}{e} \right) \cdot Q = O(1)$$

and depends at most on  $a, b, d_1, d'_1, r_1, r_2$  and the residue classes of  $u$  and  $v$  modulo  $2^5 3^4$ . Note that the dependence on  $d'_2$  is really a dependence on  $r_2$  by (5.15), and the dependence on  $f_2, f_3$  is one on  $r_1 r_2 = q$ . Moreover, we write

$$2^{g_2} 3^{g_3} \equiv A \overline{B} \pmod{e_0},$$

where  $(A, B) = 1$  and  $A$  and  $B$  are integers of the form  $2^{g'_2} 3^{g'_3}$  and  $2^{g''_2} 3^{g''_3}$ , respectively, and set

$$a_1 := \frac{Aa}{(Aa, Bbu^2)}, \quad b_1 := \frac{Bbu^2}{(Aa, Bbu^2)}. \quad (5.20)$$

We note that  $(a_1, b_1) = 1$ . Now the exponential term takes the shape

$$e \left( \frac{-2^{g_2} 3^{g_3} \cdot a \cdot \overline{b d_2^*} \cdot \overline{u}^2 v^3}{e_0} \right) = e \left( \frac{-a_1 \cdot \overline{b_1 d_2^*} \cdot v^3}{e_0} \right).$$

Using (5.8), (5.11), (5.16), (5.17) and (5.18), we have

$$\mathcal{E}_{f_2, f_3}(m\overline{r_1}, n\overline{r_1}; r_2) = d'_2 \mathcal{E}_{f_2, f_3}(\overline{r_1} d_1 d'_2 u, \overline{r_1} d'_1 v; \text{rad}(r_2)) = d'_2 \mathcal{E}_{f_2, f_3}(\overline{e_1} d'_2 u, \overline{e'_1} v; \text{rad}(r_2)).$$

We further note that  $d''_2 e_2 = \text{rad}(r_2)$ . Since  $(d''_2, e_2) = 1$ , we can use (4.7) again to factorise the last exponential sum as

$$\mathcal{E}_{f_2, f_3}(\overline{e_1} d'_2 u, \overline{e'_1} v; \text{rad}(r_2)) = \mathcal{E}_{f_2, f_3}(\overline{e_1} u, \overline{e'_1 d''_2} v; e_2) \mathcal{E}_{f_2, f_3}(0, \overline{e'_1 e_2} v; d''_2).$$

Combining everything, we obtain

$$\begin{aligned} \mathcal{S}_1 &= \frac{XY}{q^2} \sum_{\substack{(6r_1, r_2)=1 \\ r_1 r_2 = q}} \sum_{\substack{d_1 | r_1, d_2 | r_2 \\ \text{rad}(\hat{r}_1)^2 | e_1 \\ \mu^2(e_2)=1}} \sum'_{d'_1 | r_1} d'_2 \sqrt{r_1 d_1} \\ &\times \sum_{\substack{u \neq 0 \\ (u, e_1 e_2)=1}} e\left(-\frac{2^{f_2} d_1 d_2 u x_0}{q}\right) \Gamma\left(\frac{2^{f_2} d_1 d_2 u X}{q}\right) S(d_1, d'_1, d_2, r_1, r_2, u), \end{aligned} \quad (5.21)$$

where the  $'$  attached to the third summation symbol on the right-hand side encodes the condition that (5.12) holds and  $S(u) = S(d_1, d'_1, d_2, r_1, r_2, u)$  is given by

$$\begin{aligned} S(u) &:= \sum_{(v, e'_1)=1} e\left(-\frac{3^{f_3} d'_1 d'_2 v y_0}{q}\right) \Gamma\left(\frac{3^{f_3} d'_1 d'_2 v Y}{q}\right) e\left(\frac{-a_1 \cdot \overline{b_1 d_2^*} \cdot v^3}{e_0}\right) \left(\frac{v}{e}\right) \\ &\times \mathcal{E}_{f_2, f_3}(\overline{e_1} u, \overline{e'_1 d_2''} v; e_2) \mathcal{E}_{f_2, f_3}(0, \overline{e'_1 e_2} v; d_2'') Q'_{d_1, d'_1, r_1, r_2, u}(v^b), \end{aligned} \quad (5.22)$$

and we henceforth use  $v^b$  to denote the residue class of  $v$  modulo  $2^5 3^4$ . We have further dropped the dependency of  $Q'$  on  $a$  and  $b$  since these are treated as fixed integers.

Now our strategy is to utilise the cancellation in the sum over  $v$  coming from the exponential term. The main obstacle is that in the generic case, the denominator  $e_0$  is very large compared to the length of the sum over  $v$  (note that the sum over  $v$  can be freely truncated at  $q^{1+\varepsilon}/(d'_1 d'_2 Y)$  since  $\Gamma$  has rapid decay). To reduce the size of the denominator in the exponential term, we flip the numerator and denominator by means of the identity (1.6), which gives

$$e\left(\frac{-a_1 \cdot \overline{b_1 d_2^*} \cdot v^3}{e_0}\right) = e\left(\frac{-a_1 v^3}{b_1 e_0 d_2^*}\right) \cdot e\left(\frac{a_1 \overline{e_0} v^3}{b_1 d_2^*}\right),$$

where  $\overline{e_0}$  is the multiplicative inverse of  $e_0$  modulo  $b_1 d_2^*$ . The first factor on the right-hand side will turn out to be a slowly oscillating weight function. The cancellation in the sum over  $v$  will come from the second factor. The sum in (5.22) now takes the form

$$\begin{aligned} S(u) &= \sum_{(v, e'_1)=1} \Gamma\left(\frac{3^{f_3} d'_1 d'_2 v Y}{q}\right) e\left(-\frac{3^{f_3} d'_1 d'_2 v y_0}{q}\right) e\left(\frac{-a_1 v^3}{b_1 e_0 d_2^*}\right) e\left(\frac{a_1 \overline{e_0} v^3}{b_1 d_2^*}\right) \\ &\times \left(\frac{v}{e}\right) \mathcal{E}_{f_2, f_3}(\overline{e_1} u, \overline{e'_1 d_2''} v; e_2) \mathcal{E}_{f_2, f_3}(0, \overline{e'_1 e_2} v; d_2'') Q'_{d_1, d'_1, r_1, r_2, u}(v^b). \end{aligned} \quad (5.23)$$

The idea is now to write this as a short sum of complete exponential sums to modulus  $2^5 3^4 b_1 d_2^* e$ . This will be done in the next section using the Poisson summation formula.

## 6. Second application of Poisson summation

We begin by removing the coprimality condition  $(v, e'_1) = 1$  in (5.23) using Möbius inversion. This gives

$$S(u) = \sum_{\substack{t | e'_1 \\ (t, e)=1}} \mu(t) \left(\frac{t}{e}\right) S(u, t), \quad (6.1)$$



where  $S(u, t) = S(d_1, d'_1, d_2, r_1, r_2, u, t)$  is given by

$$S(u, t) = \sum_v \Gamma \left( \frac{3^{f_3} d'_1 d'_2 t v Y}{q} \right) e \left( -\frac{3^{f_3} d'_1 d'_2 t v y_0}{q} \right) e \left( \frac{-a_1 t^3 v^3}{b_1 e_0 d_2^*} \right) e \left( \frac{a_1 \bar{e}_0 t^3 v^3}{b_1 d_2^*} \right) \times \left( \frac{v}{e} \right) \mathcal{E}_{f_2, f_3}(\bar{e}_1 u, \bar{e}'_1 d''_2 t v; e_2) \mathcal{E}_{f_2, f_3}(0, \bar{e}'_1 e_2 t v; d''_2) Q''(v^\flat), \quad (6.2)$$

with

$$Q''(v^\flat) = Q''_{d_1, d'_1, r_1, r_2, u, t}(v^\flat) := Q'_{d_1, d'_1, r_1, r_2, u}(t^\flat v^\flat).$$

To simplify the notations, we define a function  $\Psi : \mathbb{R} \rightarrow \mathbb{C}$  by

$$\Psi \left( \frac{3^{f_3} d'_1 d'_2 t Y}{q} \cdot v \right) := \Gamma \left( \frac{3^{f_3} d'_1 d'_2 t Y}{q} \cdot v \right) e \left( -\frac{3^{f_3} d'_1 d'_2 t y_0}{q} \cdot v \right) e \left( \frac{-a_1 t^3}{b_1 e_0 d_2^*} \cdot v^3 \right).$$

Equivalently, if  $C := 2^{l_2-3j_2} 3^{l_3-3j_3-3f_3}$ , then

$$\begin{aligned} \Psi(z) &= \Gamma(z) e \left( -\frac{3^{f_3} d'_1 d'_2 t y_0}{q} \cdot \frac{q}{3^{f_3} d'_1 d'_2 t Y} \cdot z \right) e \left( -\frac{a_1 t^3}{b_1 e_0 d_2^*} \cdot \left( \frac{q}{3^{f_3} d'_1 d'_2 t Y} \right)^3 \cdot z^3 \right) \\ &= \Gamma(z) e \left( -\frac{y_0}{Y} \cdot z \right) e \left( -C \cdot \frac{a_1 q^2}{b_1 d_1^2 d_2^2 Y^3} \cdot z^3 \right), \end{aligned} \quad (6.3)$$

where in the last equation we have used (5.8), (5.10), (5.12), (5.14), (5.19) and  $q = r_1 r_2$ . We note that  $\Psi$  depends on  $d_1, d'_1, d_2, r_1, r_2, u, y_0$  and  $Y$ . We may now write

$$S(u, t) = \sum_v \Psi \left( \frac{3^{f_3} d'_1 d'_2 t Y}{q} \cdot v \right) e \left( \frac{a_1 \bar{e}_0 t^3 v^3}{b_1 d_2^*} \right) \left( \frac{v}{e} \right) \times \mathcal{E}_{f_2, f_3}(\bar{e}_1 u, \bar{e}'_1 d''_2 t v; e_2) \mathcal{E}_{f_2, f_3}(0, \bar{e}'_1 e_2 t v; d''_2) Q''(v^\flat).$$

Let  $D := 2^5 3^4$ . We shall split the summation over  $v$  into residue classes modulo  $D b_1 d_2^* e$ . We note that  $\text{rad}(r_2) \mid d_2^*$  by (5.19). Hence we get

$$\begin{aligned} S(u, t) &= \sum_{k \bmod D b_1 d_2^* e} e \left( \frac{a_1 \bar{e}_0 t^3 k^3}{b_1 d_2^*} \right) \left( \frac{k}{e} \right) \mathcal{E}_{f_2, f_3}(\bar{e}_1 u, \bar{e}'_1 d''_2 t k; e_2) \mathcal{E}_{f_2, f_3}(0, \bar{e}'_1 e_2 t k; d''_2) Q''(k^\flat) \\ &\times \sum_{v \equiv k \bmod D b_1 d_2^* e} \Psi \left( \frac{3^{f_3} d'_1 d'_2 t Y}{q} \cdot v \right), \end{aligned}$$

where  $k^\flat$  denotes the residue class of  $k$  modulo  $D$ . We detect this residue class using additive characters, getting

$$\begin{aligned} S(u, t) &= \frac{1}{D} \sum_{x=1}^D \sum_{y=1}^D e \left( \frac{xy}{D} \right) Q''(y) \sum_{k \bmod D b_1 d_2^* e} e \left( \frac{a_1 \bar{e}_0 t^3 k^3}{b_1 d_2^*} \right) \left( \frac{k}{e} \right) \mathcal{E}_{f_2, f_3}(\bar{e}_1 u, \bar{e}'_1 d''_2 t k; e_2) \\ &\times \mathcal{E}_{f_2, f_3}(0, \bar{e}'_1 e_2 t k; d''_2) e \left( -\frac{xk}{D} \right) \sum_{v \equiv k \bmod D b_1 d_2^* e} \Psi \left( \frac{3^{f_3} d'_1 d'_2 t Y}{q} \cdot v \right). \end{aligned} \quad (6.4)$$

Applying Poisson summation to the inner-most sum over  $v$  after a linear change of variables, we see that it is

$$\begin{aligned} &= \frac{q}{3^{f_3} D b_1 d_2^* e d'_1 d'_2 t Y} \sum_{h \in \mathbb{Z}} e \left( \frac{kh}{D b_1 d_2^* e} \right) \hat{\Psi} \left( \frac{q}{3^{f_3} D b_1 d_2^* e d'_1 d'_2 t Y} \cdot h \right) \\ &= \frac{r_1}{3^{f_3} D b_1 e d'_1 d_2^{\prime 2} t Y} \sum_{h \in \mathbb{Z}} e \left( \frac{kh}{D b_1 d_2^* e} \right) \hat{\Psi} \left( \frac{r_1}{3^{f_3} D b_1 e d'_1 d_2^{\prime 2} t Y} \cdot h \right), \end{aligned} \quad (6.5)$$

where in the last equation, we have used (5.15), (5.19) and  $q = r_1 r_2$ .

Combining (6.4) and (6.5) we get

$$S(u, t) = \frac{r_1}{3^{f_3} D^2 b_1 e d_1' d_2''^2 t Y} \sum_{x=1}^D \sum_{y=1}^D e\left(\frac{xy}{D}\right) Q''(y) \sum_{h \in \mathbb{Z}} \hat{\Psi}\left(\frac{r_1}{3^{f_3} D b_1 e d_1' d_2''^2 t Y} \cdot h\right) F_x(h), \quad (6.6)$$

where  $F_x(h)$  is the complete exponential sum

$$\begin{aligned} F_x(h) := & \sum_{k \bmod Db_1 d_2^* e} e\left(\frac{kh}{Db_1 d_2^* e}\right) e\left(\frac{a_1 \bar{e}_0 t^3 k^3}{b_1 d_2^*}\right) \left(\frac{k}{e}\right) e\left(-\frac{xk}{D}\right) \\ & \times \mathcal{E}_{f_2, f_3}(\bar{e}_1 u, \bar{e}_1' d_2'' t k; e_2) \mathcal{E}_{f_2, f_3}(0, \bar{e}_1' e_2 t k; d_2''). \end{aligned}$$

We now split the exponential sum  $F_x(h)$  into parts and estimate them. We observe that

$$Db_1 d_2^* e = Db_1 d_2''^3 e_2 e$$

by (5.19) and a little thought reveals that  $Db_1 d_2''^3$ ,  $e_2$ , and  $e$  are pairwise coprime. Therefore we may write  $k \bmod Db_1 d_2^* e$  in the form  $k = \alpha \cdot e_2 e + \beta \cdot Db_1 d_2''^3 e_2 + \gamma \cdot Db_1 d_2''^3 e$ , where  $\alpha$  runs over residue classes mod  $Db_1 d_2''^3$ ,  $\beta$  runs over residue classes mod  $e$ , and  $\gamma$  runs over residue classes mod  $e_2$ . It follows that

$$F_x(h) = G_1(h) G_2(h) G_3(h), \quad (6.7)$$

where

$$\begin{aligned} G_1(h) &:= \sum_{\alpha \bmod Db_1 d_2''^3} e\left(\frac{Da_1 \bar{e}_0 t^3 e^3 e_2^2 \alpha^3 + h\alpha}{Db_1 d_2''^3}\right) e\left(-\frac{x e_2 e \alpha}{D}\right) \mathcal{E}_{f_2, f_3}(0, \bar{e}_1' t e \alpha; d_2''), \\ G_2(h) &:= \left(\frac{Db_1 \text{rad}(r_2)}{e}\right) \sum_{\beta \bmod e} e\left(\frac{h\beta}{e}\right) \cdot \left(\frac{\beta}{e}\right), \\ G_3(h) &:= \sum_{\gamma \bmod e_2} e\left(\frac{h\gamma}{e_2}\right) e\left(\frac{D^3 a_1 \bar{e}_0 t^3 e^3 b_1^2 d_2''^6 \gamma^3}{e_2}\right) \mathcal{E}_{f_2, f_3}(\bar{e}_1 u, D \bar{e}_1' t b_1 d_2''^2 e \gamma; e_2). \end{aligned}$$

We note that only  $G_1(h)$  depends on  $x$ . Our next task is to provide satisfactory upper bounds for the modulus of these exponential sums.

Beginning with  $G_2(h)$  we see that this is a Gauss sum with a quadratic character. Since the modulus  $e$  is squarefree this character is primitive and it follows that

$$|G_2(h)| \leq e^{1/2}. \quad (6.8)$$

Moreover, if  $h = 0$  then we have

$$G_2(0) = \begin{cases} 0 & \text{if } e > 1, \\ 1 & \text{if } e = 1. \end{cases} \quad (6.9)$$

Turning to  $G_3(h)$ , for which we use  $(\bar{e}_1 u, e_2) = 1$  and Lemma 4.6, we deduce that

$$|G_3(h)| \leq \sum_{\gamma \bmod e_2} |\mathcal{E}_{f_2, f_3}(\bar{e}_1 u, D \bar{e}_1' t b_1 d_2''^2 e \gamma; e_2)| \ll e_2^{3/2} 2^{\omega(e_2)}. \quad (6.10)$$

The treatment of  $G_1(h)$  is more taxing and we will need to introduce some more notation. For a natural number  $n$  let  $\rho(n)$  be defined multiplicatively by

$$\rho(p^\alpha) := p^{f(\alpha)}, \quad (6.11)$$

where

$$f(\alpha) := \begin{cases} 1/2 & \text{if } \alpha = 1, \\ 1 & \text{if } \alpha = 2 \text{ or } \alpha = 3, \\ 5/4 & \text{if } \alpha = 4, \\ \alpha/4 & \text{if } \alpha \geq 5. \end{cases} \quad (6.12)$$

We note that  $\rho(n) \leq \sqrt{n}$  for every  $n \in \mathbb{N}$ . Furthermore, for any  $\alpha \leq \beta$  one easily checks that

$$\alpha - f(\alpha) \leq \beta - f(\beta), \quad f(\alpha) \leq f(\beta), \quad f(\alpha + \beta) \leq f(\alpha) + f(\beta).$$

Hence it follows that  $m/\rho(m) \leq n/\rho(n)$  and  $\rho(m) \leq \rho(n)$  if  $m \mid n$ , and furthermore

$$\rho(n_1 n_2) \leq \rho(n_1) \rho(n_2),$$

for all  $n_1, n_2$ . The estimation of  $G_1(h)$  is the object of the following result.

**Lemma 6.1.** — *We have*

$$G_1(h) \ll \begin{cases} d_2''^{7/2} b_1^{1/2} (h, b_1)^{1/2} 2^{\omega(b_1)} (4 + \varepsilon)^{\omega(d_2'')} & \text{if } h \neq 0, \\ a^{1/2} b_1 2^{\omega(b_1)} d_2''^3 / \rho(bu^2) & \text{if } h = 0. \end{cases}$$

*Proof.* — To deal with  $G_1(h)$  we write  $\alpha = \alpha_1 + Dd_2''\alpha_2$ , where  $\alpha_1$  runs over the residue classes mod  $Dd_2''$ , and  $\alpha_2$  runs over the residue classes mod  $b_1 d_2''^2$ . This gives

$$\begin{aligned} G_1(h) &= \sum_{\alpha_1 \bmod Dd_2''} e \left( \frac{Da_1 \overline{e_0} t^3 e^3 e_2^2 \alpha_1^3 + h \alpha_1}{Db_1 d_2''^3} \right) e \left( -\frac{x e_2 e \alpha_1}{D} \right) \\ &\quad \times \mathcal{E}_{f_2, f_3}(0, \overline{e_1} t e \alpha_1; d_2'') G_1(h, \alpha_1), \end{aligned} \quad (6.13)$$

where

$$G_1(h, \alpha_1) := \sum_{\alpha_2 \bmod b_1 d_2''^2} e \left( \frac{P_{h, \alpha_1}(\alpha_2)}{b_1 d_2''^2} \right)$$

and

$$P_{h, \alpha_1}(X) := Da_1 \overline{e_0} t^3 e^3 e_2^2 \cdot \frac{(\alpha_1 + Dd_2''X)^3 - \alpha_1^3}{Dd_2''} + hX.$$

Using  $(\overline{e_1} t e, d_2'') = 1$ , we deduce from Lemma 4.6 that

$$|\mathcal{E}(0, \overline{e_1} t e \alpha_1; d_2'')| \ll (\alpha_1, d_2'')^{1/2} d_2''^{1/2} 2^{\omega(d_2'')}. \quad (6.14)$$

Let us begin by considering the case  $h \neq 0$ , to deal with which we call upon the general upper bound for complete exponential sums presented in Lemma 4.1. We wish to apply this with  $Q = b_1 d_2''^2$  and  $g(X) = P_{h, \alpha_1}(X)$ . We note that  $g'(X) = 3Da_1 \overline{e_0} t^3 e^3 e_2^2 (\alpha_1 + Dd_2''X)^2 + h$ . Thus we have

$$n = 2, \quad \eta = 1, \quad \Delta = 4hA, \quad A = 3D^3 a_1 \overline{e_0} t^3 e^3 e_2^2 d_2''^2,$$

and

$$(\Delta, Q) = d_2''^2 (12D^3 a_1 \overline{e_0} t^3 e^3 e_2^2 h, b_1).$$

Recalling that  $D = 2^5 3^4$  and that  $t$  can be assumed to be square-free with  $t/(t, 6)$  coprime to  $b_1$ , so  $(t^3, b_1) \mid 2^3 3^3$ , we readily conclude that  $(\Delta, Q) \ll d_2''^2 (h, b_1)$ . Hence

$$G_1(h, \alpha_1) \ll b_1^{1/2} (h, b_1)^{1/2} d_2''^2 2^{\omega(b_1 d_2''^2)}.$$

Combining this with (3.17), (6.13) and (6.14) we obtain the bound for  $G_1(h)$  in the statement of the lemma.

Suppose now that  $h = 0$  and let  $\delta := Da_1\bar{e}_0t^3e^3e_2^2$ . Then we have

$$G_1(0, \alpha_1) = \sum_{\alpha_2 \bmod b_1 d_2''^2} e \left( \frac{\delta D^2 \alpha_2^3}{b_1} + \frac{3\delta D \alpha_1 \alpha_2^2}{b_1 d_2''^2} + \frac{3\delta \alpha_1^2 \alpha_2}{b_1 d_2''^2} \right).$$

We may write  $\alpha_2 = x_1 + x_2 \cdot b_1 d_2''$ , with  $x_1 \in \mathbb{Z}/b_1 d_2'' \mathbb{Z}$  and  $x_2 \in \mathbb{Z}/d_2'' \mathbb{Z}$ . It follows that

$$G_1(0, \alpha_1) = \sum_{x_1 \bmod b_1 d_2''} e \left( \frac{\delta D^2 x_1^3}{b_1} + \frac{3\delta D \alpha_1 x_1^2}{b_1 d_2''} + \frac{3\delta \alpha_1^2 x_1}{b_1 d_2''^2} \right) \sum_{x_2 \bmod d_2''} e \left( \frac{3\delta \alpha_1^2 x_2}{d_2''} \right).$$

Using the facts that  $(3\delta, d_2'') = 1$  and  $d_2''$  is square-free, we deduce that the innermost sum on the right-hand side of the above equation is 0 unless  $\alpha_1 \equiv 0 \bmod d_2''$ . Hence we have

$$G_1(0, \alpha_1) = 0 \quad \text{if } \alpha_1 \not\equiv 0 \bmod d_2'' \quad (6.15)$$

and

$$\begin{aligned} G_1(0, j d_2'') &= d_2''^2 \sum_{x \bmod b_1} e \left( \frac{\delta D^2 x^3 + 3\delta D j x^2 + 3\delta j^2 x}{b_1} \right) \\ &= d_2''^2 \sum_{x \bmod b_1} e \left( \frac{\delta ((Dx + j)^3 - j^3)}{D b_1} \right) \\ &= \frac{d_2''^2}{D} \sum_{x \bmod D b_1} e \left( \frac{\delta ((Dx + j)^3 - j^3)}{D b_1} \right). \end{aligned}$$

Let  $E := (\delta, D b_1)$ ,  $\delta' := \delta/E$  and  $b_1' := D b_1/E$ . Then it follows that

$$G_1(0, j d_2'') = \frac{E d_2''^2}{D} \sum_{x=1}^{b_1'} e \left( \frac{\delta' ((Dx + j)^3 - j^3)}{b_1'} \right). \quad (6.16)$$

As above we have  $E = O(1)$ .

In the following, we want to bound the sum on the right-hand side of (6.16). To this end we consider in general terms exponential sums of the form

$$G(c, r; Q) = \sum_{x=1}^Q e \left( \frac{c ((D r x + j)^3 - j^3) / r}{Q} \right).$$

A standard calculation shows that these exponential sums are multiplicative in the sense that

$$G(c, r; Q) = G(c, r Q_2; Q_1) G(c, r Q_1; Q_2) \quad (6.17)$$

if  $Q = Q_1 Q_2$  with  $(Q_1, Q_2) = 1$ . Moreover, we note that if  $(r, Q) = 1$ , then

$$G(c, r; Q) = \sum_{x=1}^Q e \left( \frac{c \bar{r} ((D r x + j)^3 - j^3)}{Q} \right). \quad (6.18)$$

If in addition  $(D, Q) = 1$ , then  $D r x + j$  runs through all residue classes modulo  $Q$  as  $x$  runs through all of them, and therefore it follows that

$$G(c, r; Q) = e \left( -\frac{c \bar{r} j^3}{Q} \right) \sum_{x=1}^Q e \left( \frac{c \bar{r} x^3}{Q} \right). \quad (6.19)$$

Hence  $G(c, r; Q)$  is essentially a cubic Gauss sum in this case.

We claim the bound

$$|G(c, r; Q)| \leq (Q, 9D^6) \cdot \frac{Q}{\rho(Q)} \cdot 2^{\omega(Q)}, \quad (6.20)$$

if  $(cr, Q) = 1$ , where  $\rho(Q)$  is given multiplicatively by (6.11) and (6.12). By (6.17), to prove this it suffices to establish this bound for  $Q$  a prime power. Suppose that  $Q = p^\alpha$  with  $p$  prime and  $\alpha \geq 1$ . Clearly (6.20) follows from the Weil bound if  $\alpha = 1$  and  $p > 3$ . It follows from Lemma 4.1 with  $n = 2, \eta = 2$  and  $\Delta = (3cD^3r^2)^2$  if  $\alpha \geq 5$ . Next we note that

$$\sum_{x=1}^{p^2} e\left(\frac{c\bar{r}x^3}{p^2}\right) = p + E(c\bar{r}, 0; p^2), \quad \sum_{x=1}^{p^3} e\left(\frac{c\bar{r}x^3}{p^3}\right) = p^2 + E(c\bar{r}, 0; p^3),$$

in the notation of (4.1). Moreover, by Lemma 4.4(ii), we have  $E(c\bar{r}, 0; p^2) = 0 = E(c\bar{r}, 0; p^3)$  if  $p > 3$ . Thus (6.20) follows from (6.19) if  $\alpha = 2, 3$  and  $p > 3$ . When  $\alpha = 4$  and  $p > 3$  we deduce from Lemma 4.4(ii) that

$$\sum_{x=1}^{p^4} e\left(\frac{c\bar{r}x^3}{p^4}\right) = \sum_{\substack{x=1 \\ p|x}}^{p^4} e\left(\frac{c\bar{r}x^3}{p^4}\right) + E(c\bar{r}, 0, p^4) = p^2 \sum_{x=1}^p e\left(\frac{c\bar{r}x^3}{p}\right),$$

which has modulus at most  $2p^{5/2}$  by the Weil estimate if  $p > 3$ . Since (6.20) follows trivially if  $1 \leq \alpha \leq 4$  and  $p = 2, 3$ , this therefore completes its proof.

Employing (6.20), the definition of  $b_1$  in (5.20) and the facts about the function  $\rho(Q)$  stated above before the lemma, we deduce that

$$G(\delta', 1; b'_1) \ll \frac{b'_1 2^{\omega(b'_1)}}{\rho(b'_1)} \ll \frac{b_1 2^{\omega(b_1)}}{\rho(b_1)} \ll \frac{b_1 2^{\omega(b_1)} \rho(Aa)}{\rho(bu^2)} \ll \frac{b_1 2^{\omega(b_1)} a^{1/2}}{\rho(bu^2)}.$$

Therefore

$$G_1(0, \alpha) = \frac{Ed_2''^2}{D} \cdot G(\delta', 1; b'_1) \ll \frac{a^{1/2} b_1 2^{\omega(b_1)} d_2''^2}{\rho(bu^2)}.$$

Combining this with (6.13) and (6.15), and using the trivial estimate  $\mathcal{E}(0, 0; d_2'') \leq d_2''$ , we therefore get the second part of the lemma.  $\square$

Recall that  $d_2'' e_2 = \text{rad}(r_2)$  and  $d_1' \gg d_1$ . In particular  $2^{\omega(e_2)}(4 + \varepsilon)^{\omega(d_2'')} \leq (4 + \varepsilon)^{\omega(r_2)}$ . Then from (6.6)–(6.10) and Lemma 6.1 we obtain

$$S(u, t) \ll A_0(u, t) + A_1(u, t), \quad (6.21)$$

in (6.2), where

$$A_0(u, t) := \frac{a^{1/2} r_1 \cdot \text{rad}(r_2) e_2^{1/2} 2^{\omega(b_1) + \omega(e_2)}}{d_1 \rho(bu^2) t Y} \cdot |\hat{\Psi}(0)|, \quad (6.22)$$

$$A_1(u, t) := \frac{r_1 \cdot \text{rad}(r_2)^{3/2} 2^{\omega(b_1)} (4 + \varepsilon)^{\omega(r_2)}}{d_1 e^{1/2} b_1^{1/2} t Y} \sum_{h \neq 0} \left| \hat{\Psi} \left( \frac{r_1}{3^{f_2} D b_1 e d_1' d_2''^2 t Y} \cdot h \right) \right| (h, b_1)^{1/2}. \quad (6.23)$$

Let

$$\alpha := \frac{r_1}{3^{f_2} D b_1 e d_1' d_2''^2 t Y}, \quad \beta := \frac{C a_1 q^2}{b_1 d_1^2 d_2^2 Y^3}, \quad \gamma := \frac{y_0}{Y}. \quad (6.24)$$

We now turn to the estimation of  $|\hat{\Psi}(z)|$ . By (6.3) we have

$$\hat{\Psi}(z) = F(z + \gamma, \beta), \quad (6.25)$$

where  $F$  is given by (3.3). We note that  $\alpha > 0$  and  $\beta > 0$  since we assumed at the outset that  $a, b > 0$ . Hence we are free to apply the estimates §3.2. Our goal is to bound  $|\hat{\Psi}(0)|$  in (6.22) and the series

$$\sum_{h \neq 0} |\hat{\Psi}(\alpha h)| (h, b_1)^{1/2} = \sum_{h \neq 0} |F(\gamma + \alpha h, \beta)| (h, b_1)^{1/2}$$

in (6.23). We shall split the latter into three ranges.

By (3.4) and (3.17), we have

$$\sum_{1 \leq |h| \leq 2|\gamma|/\alpha} |F(\gamma + \alpha h, \beta)|(h, b_1)^{1/2} \ll \sum_{1 \leq |h| \leq 2|\gamma|/\alpha} (h, b_1)^{1/2} \ll \frac{|\gamma|}{\alpha} (1 + \varepsilon)^{\omega(b_1)}. \quad (6.26)$$

Next let  $\kappa$  be any positive real number and let  $P := 2abqXY$ . Using  $|\gamma + \alpha h| \gg \alpha|h|$  for  $|h| > 2|\gamma|/\alpha$  and (3.13), we combine (3.17) with partial summation to obtain

$$\begin{aligned} \sum_{2|\gamma|/\alpha < |h| \leq P^\kappa} |F(\gamma + \alpha h, \beta)|(h, b_1)^{1/2} &\ll \sum_{1 \leq |h| \leq P^\kappa} \frac{(h, b_1)^{1/2}}{\alpha|h|} \\ &\quad + \sum_{h \geq 1} \frac{\exp(-\pi\alpha h/(3\beta))}{(\alpha\beta h)^{1/4}} (h, b_1)^{1/2} \\ &\ll \left( \frac{\log P}{\alpha} + \frac{\beta^{1/2}}{\alpha} \right) (1 + \varepsilon)^{\omega(b_1)}. \end{aligned} \quad (6.27)$$

Let us give a bit more explanation about how the second term in the last line arises. The term  $\exp(-\pi\alpha h/(3\beta))$  is negligible if  $h$  is much larger than  $\beta/\alpha$ . Moreover (3.17) ensures that the term  $(h, b_1)^{1/2}$  has average order  $O((1 + \varepsilon)^{\omega(b_1)})$ . Using a dyadic summation one readily concludes that the sum in question is bounded by the sum over  $1 \leq h \ll \beta/\alpha$  of the term  $(1 + \varepsilon)^{\omega(b_1)}/(\alpha\beta h)^{1/4}$ , which is  $\ll (1 + \varepsilon)^{\omega(b_1)}\beta^{1/2}/\alpha$ .

Finally if  $\kappa > 0$  is large enough, then we infer from (3.14) that

$$\sum_{|h| > P^\kappa} |F(\gamma + \alpha h, \beta)|(h, b_1)^{1/2} \ll P^{-1000}. \quad (6.28)$$

Combining (6.26), (6.27) and (6.28), we get

$$\sum_{h \neq 0} |\hat{\Psi}(\alpha h)|(h, b_1)^{1/2} \ll \frac{\log P + |\gamma| + \beta^{1/2}}{\alpha} \cdot (1 + \varepsilon)^{\omega(b_1)}. \quad (6.29)$$

From (5.16), (6.24),  $d'_1 \ll d_1$  and  $q = r_1 r_2$ , we therefore deduce that

$$\begin{aligned} &\sum_{h \neq 0} |\hat{\Psi}(\alpha h)|(h, b_1)^{1/2} \\ &\ll \left( \frac{b_1 e d_1 d_2^2 \cdot \text{rad}(r_2)^2 t \cdot Y_0 \log P}{r_1 r_2^2} + \frac{(a_1 b_1)^{1/2} e d_2 \cdot \text{rad}(r_2)^2 t}{r_2 Y^{1/2}} \right) (1 + \varepsilon)^{\omega(b_1)}, \end{aligned} \quad (6.30)$$

where

$$Y_0 := |y_0| + Y. \quad (6.31)$$

We close this section by recording an upper bound for  $\hat{\Psi}(0)$ . Using (3.4) and (3.13), we have the bound

$$\hat{\Psi}(0) \ll \min \left\{ 1, \frac{1}{|\gamma|^{1/4} \beta^{1/4}} + \frac{1}{|\gamma|} \right\} \ll \frac{1}{(|\gamma| + 1)^{1/4} \beta^{1/4}} + \frac{1}{|\gamma|}.$$

Combining this with (5.20) and (6.24), we therefore derive the estimate

$$\hat{\Psi}(0) \ll \frac{b^{1/4} |u|^{1/2} d_1^{1/2} d_2^{1/2} Y}{a^{1/4} q^{1/2} Y_0^{1/4}} + \frac{Y}{Y_0}, \quad (6.32)$$

where  $Y_0$  is given in (6.31).

## 7. Conclusion of the proof of Theorem 1.2

We now derive our final estimate for  $\mathcal{S}_1$ , before combining it with the contents of §5 to complete the asymptotic formula for  $\mathcal{S}$ . This will in turn lead us to the statement of Theorem 1.2.

Combining (6.1), (6.21), (6.30), (6.32),  $q = r_1 r_2$ ,  $d_1 \gg d'_1 \geq d_1$  and  $e_i = r_i/d_i$ , we get

$$S(u) \ll A_0(u) + A_1(u),$$

where

$$A_0(u) := \left( \frac{a^{1/4} b^{1/4} r_1^{1/2} \operatorname{rad}(r_2) |u|^{1/2}}{d_1^{1/2} \rho(bu^2) Y_0^{1/4}} + \frac{a^{1/2} r_1 \cdot \operatorname{rad}(r_2) r_2^{1/2}}{d_1 d_2^{1/2} \rho(bu^2) Y_0} \right) 2^{\omega(b_1) + \omega(e_2)} \sigma_{-1}(e_1), \quad (7.1)$$

$$A_1(u) := \left( \frac{b_1^{1/2} e^{1/2} d_2^2 \cdot \operatorname{rad}(r_2)^{7/2} \cdot Y_0 \log P}{r_2^2 Y} + \frac{a_1^{1/2} e^{1/2} d_2 r_1 \cdot \operatorname{rad}(r_2)^{7/2}}{d_1 r_2 Y^{3/2}} \right) \times (2 + \varepsilon)^{\omega(b_1)} 2^{\omega(e_1)} (4 + \varepsilon)^{\omega(r_2)}. \quad (7.2)$$

To derive an estimate for  $\mathcal{S}_1$ , we have to sum the above expressions over  $u$  as dictated by (5.21). Let

$$I_q := \sum_{\substack{u \in \mathbb{Z} \setminus \{0\} \\ (u, e_1 e_2) = 1}} \Gamma \left( \frac{2^{f_2} d_1 d_2 u X}{q} \right) |S(u)|.$$

For any  $M > 0$  let us consider the pair of sums

$$\Sigma_i(M) := \sum_{\substack{0 < |u| \leq M \\ (u, e_1 e_2) = 1}} A_i(u),$$

for  $i = 0, 1$ , with  $A_0(u), A_1(u)$  given by (7.1) and (7.2), respectively. Estimating these is the object of the following pair of results.

**Lemma 7.1.** — *We have*

$$\begin{aligned} \Sigma_0(M) &\ll \sigma_0(M) := (\log(2 + M))^2 \left( \frac{a^{1/2} \operatorname{rad}(r_2) r_1 r_2^{1/2}}{\rho(b) d_1 d_2^{1/2} Y_0} + \frac{a^{1/4} b^{1/4} \operatorname{rad}(r_2) r_1^{1/2} M^{1/2}}{\rho(b) d_1^{1/2} Y_0^{1/4}} \right) \\ &\times (\log q)^\varepsilon (2 + \varepsilon)^{\omega(b)} 2^{\omega(e_2)}. \end{aligned}$$

*Proof.* — Recall that  $d'_2 = \operatorname{rad}(r_2)/e_2$  and  $e_2 = r_2/d_2$ . Applying the bound  $\sigma_{-1}(e_1) \ll (\log q)^\varepsilon$  we insert our expression for  $A_0(u)$  to see that

$$\Sigma_0(M) \leq (\log q)^\varepsilon 2^{\omega(e_2)} \left( \frac{a^{1/2} \operatorname{rad}(r_2) r_1 r_2^{1/2}}{d_1 d_2^{1/2} Y_0} + \frac{a^{1/4} b^{1/4} M^{1/2} \operatorname{rad}(r_2) r_1^{1/2}}{d_1^{1/2} Y_0^{1/4}} \right) \sum_{|u| \leq M} \frac{2^{\omega(b_1)}}{\rho(bu^2)},$$

where  $C$  is given multiplicatively by (6.11) and (6.12). We claim that

$$\sum_{|u| \leq M} \frac{2^{\omega(b_1)}}{\rho(bu^2)} \ll \frac{(2 + \varepsilon)^{\omega(b)}}{\rho(b)} (\log(2 + M))^2,$$

which once inserted into the above leads to the statement of the lemma.

To establish the claim we see that

$$\sum_{|u| \leq M} \frac{2^{\omega(b_1)}}{\rho(bu^2)} \ll \frac{2^{\omega(b)}}{\rho(b)} \sum_{u=1}^M \frac{\rho(b) 2^{\omega(u)}}{\rho(bu^2)}.$$

Clearly

$$\sum_{u=1}^M \frac{\rho(b)2^{\omega(u)}}{\rho(bu^2)} \leq \prod_{\substack{p \leq M \\ p^\beta \parallel b}} \left( 1 + \sum_{k \geq 1} \frac{2p^{f(\beta)}}{p^{f(\beta+2k)}} \right) \prod_{\substack{p \leq M \\ p \nmid b}} \left( 1 + \sum_{k \geq 1} \frac{2}{p^{f(2k)}} \right) = \Pi_1 \Pi_2,$$

say. Merten's theorem yields  $\Pi_2 \ll (\log(2+M))^2$ . Moreover we have

$$\Pi_1 \leq \prod_{p^\beta \parallel b} \left( 1 + \sum_{k \geq 1} \frac{2p^{f(\beta)}}{p^{f(\beta+2k)}} \right) \ll (1+\varepsilon)^{\omega(b)}.$$

Putting these together therefore establishes the claim.  $\square$

**Lemma 7.2.** — *Let  $P = 2abqXY$ . We have*

$$\begin{aligned} \Sigma_1(M) &\ll \sigma_1(M) := \log(2+M)^{1+\varepsilon} \cdot \frac{e^{1/2} r_1 \operatorname{rad}(r_2)^{7/2}}{r_2} \left( \frac{b^{1/2} d_2^2 Y_0 \log P}{qY} \cdot M^2 + \frac{a^{1/2} d_2}{d_1 Y^{3/2}} \cdot M \right) \\ &\quad \times (2+\varepsilon)^{\omega(b)} 2^{\omega(e_1)} (4+\varepsilon)^{\omega(r_2)}. \end{aligned}$$

*Proof.* — This follows from insertion of our expression for  $A_1(u)$  into  $\Sigma_1(M)$ , using  $a_1 \leq a$  and  $b_1 \leq bu^2$  and applying (3.16).  $\square$

Since  $\Gamma$  has exponential decay, it is easily seen by a dyadic summation that Lemmas 7.1 and 7.2 imply the bound  $I_q \ll \sigma_0(M) + \sigma_1(M)$ , with

$$M := \frac{q}{d_1 d_2 X}.$$

It now follows that

$$I_q \ll (\log P)^{2+\varepsilon} \left( I_q^{(0)} + I_q^{(1)} \right),$$

where

$$\begin{aligned} I_q^{(0)} &:= \frac{\operatorname{rad}(r_2) r_1^{1/2} q^{1/2}}{\rho(b) d_1 d_2^{1/2}} \left( \frac{a^{1/2}}{Y_0} + \frac{a^{1/4} b^{1/4}}{X^{1/2} Y_0^{1/4}} \right) (2+\varepsilon)^{\omega(b)} 2^{\omega(e_2)}, \\ I_q^{(1)} &:= \frac{e^{1/2} \operatorname{rad}(r_2)^{7/2} r_1 q}{d_1^2 r_2 X} \left( \frac{b^{1/2} Y_0}{XY} + \frac{a^{1/2}}{Y^{3/2}} \right) (2+\varepsilon)^{\omega(b)} 2^{\omega(e_1)} (4+\varepsilon)^{\omega(r_2)}. \end{aligned}$$

Recalling from (5.15) that  $d_2' = r_2 / \operatorname{rad}(r_2)$ , we deduce from (5.21) that our work so far has shown that

$$\mathcal{S}_1 \ll \frac{XY}{q^2} \cdot (\log P)^{2+\varepsilon} \sum_{\substack{(r_1, r_2)=1 \\ r_1 r_2 = q}} \sum_{\substack{d_1 | r_1 \\ \operatorname{rad}(\hat{r}_1)^2 | e_1}} \sum_{\substack{d_2 | r_2 \\ \mu^2(e_2)=1}} \sum_{d_1' | r_1}' \frac{\sqrt{d_1 r_1 r_2}}{\operatorname{rad}(r_2)} \left( I_q^{(0)} + I_q^{(1)} \right).$$

Let us write  $\mathcal{S}_1^{(i)}$  for the overall contribution from the term involving  $I_q^{(i)}$  for  $i = 0, 1$ .

Beginning with the case  $i = 0$  we see that

$$\begin{aligned} \mathcal{S}_1^{(0)} &\ll \frac{XY (2+\varepsilon)^{\omega(b)} (\log P)^{2+\varepsilon}}{\rho(b) q^{1/2}} \left( \frac{a^{1/2}}{Y_0} + \frac{a^{1/4} b^{1/4}}{X^{1/2} Y_0^{1/4}} \right) \\ &\quad \times \sum_{\substack{(r_1, r_2)=1 \\ r_1 r_2 = q}} \sum_{d_1 | r_1} \sum_{d_2 | r_2} \sum_{d_1' | r_1}' \frac{2^{\omega(e_2)}}{d_1^{1/2} d_2^{1/2}}. \end{aligned}$$



Here we recall that the ' attached to the inner sum denotes the condition described in (5.12). Thus for fixed  $d_1$  the number of available  $d'_1$  is  $O(1)$ . Thus the sum over  $r_1, r_2, d_1, d_2, d'_1$  is clearly at most

$$\sum_{\substack{(r_1, r_2)=1 \\ r_1 r_2 = q}} \sigma_{-1/2}(r_1) \sigma_{-1/2}(r_2) 2^{\omega(r_2)} \ll (1 + \varepsilon)^{\omega(q)} \sum_{\substack{r_2 | q \\ (q/r_2, r_2)=1}} 2^{\omega(r_2)} \leq (3 + \varepsilon)^{\omega(q)},$$

as can be seen by checking at prime powers. On observing that  $\rho(b) \geq b^{1/4}$  we therefore conclude that

$$\mathcal{S}_1^{(0)} \ll \frac{(2 + \varepsilon)^{\omega(b)} (3 + \varepsilon)^{\omega(q)} (\log P)^{2+\varepsilon}}{b^{1/4} q^{1/2}} \left( \frac{a^{1/2} XY}{Y_0} + \frac{a^{1/4} b^{1/4} X^{1/2} Y}{Y_0^{1/4}} \right), \quad (7.3)$$

with  $Y_0 = |y_0| + Y$ .

Turning to the case  $i = 1$  we obtain

$$\mathcal{S}_1^{(1)} \ll q^{1/2} (2 + \varepsilon)^{\omega(b)} (\log P)^{2+\varepsilon} \left( \frac{b^{1/2} Y_0}{X} + \frac{a^{1/2}}{Y^{1/2}} \right) S_q$$

where

$$S_q := \sum_{\substack{(r_1, r_2)=1 \\ r_1 r_2 = q}} \sum_{\substack{d_1 | r_1 \\ \text{rad}(\hat{r}_1)^2 | e_1}} \sum_{\substack{d_2 | r_2 \\ \mu^2(e_2)=1}} \frac{e^{1/2} \text{rad}(r_2)^{5/2}}{d_1^{3/2} r_2^{3/2}} \cdot 2^{\omega(e_1)} (4 + \varepsilon)^{\omega(r_2)}.$$

Here we have carried out the summation over  $d'_1$  trivially. The latter sum is a multiplicative function in  $q$  and it will suffice to analyse it at prime powers  $q = p^\nu$  for  $\nu \in \mathbb{N}$ . Let us write  $r_1 = p^\alpha$ ,  $r_2 = p^\beta$  and  $e \leq e(e_1)$ , the square-free kernel of  $e_1$ . Then the outer summation is over  $\alpha, \beta \geq 0$  for which  $\alpha + \beta = \nu$  and  $\min\{\alpha, \beta\} = 0$ . Put  $S_{p^\nu} = T_1 + T_2$  where  $T_1$  is the contribution from  $(\alpha, \beta) = (0, \nu)$  and  $T_2$  is the contribution from  $(\alpha, \beta) = (\nu, 0)$ . An easy calculation shows that

$$T_1 = (4 + \varepsilon) p^{(5-3\nu)/2} \sum_{\substack{0 \leq \delta_2 \leq \nu \\ \nu - \delta_2 \leq 1}} 1 = (8 + \varepsilon) p^{(5-3\nu)/2}$$

and

$$T_2 = \sum_{\substack{0 \leq \delta_1 \leq \nu \\ 2 \leq \nu - \delta_1 \text{ if } p > 3}} \frac{e(p^{\nu-\delta_1})^{1/2}}{p^{3\delta_1/2}} \cdot 2^{\omega(p^{\nu-\delta_1})} \begin{cases} \leq 4, & \text{if } p \leq 3, \\ = 0, & \text{if } \nu = 1 \text{ and } p > 3, \\ = 2(T_{2,1} + T_{2,2}), & \text{if } \nu \geq 2 \text{ and } p > 3, \end{cases}$$

where

$$T_{2,1} := \sum_{\substack{0 \leq \delta_1 \leq \nu-2 \\ 2 | \nu - \delta_1}} p^{-3\delta_1/2}, \quad T_{2,2} := \sum_{\substack{0 \leq \delta_1 \leq \nu-2 \\ 2 \nmid \nu - \delta_1}} p^{(1-3\delta_1)/2}.$$

It is now clear that

$$T_{2,1} + T_{2,2} \leq p^\kappa (1 + O(1/p)),$$

where  $\kappa = 0$  if  $\nu$  is even, and  $\kappa = 1/2$  if  $\nu$  is odd. Putting this all together we conclude that

$$S_{p^\nu} \leq \begin{cases} (8 + \varepsilon)p, & \text{if } \nu = 1, \\ 2p^\kappa (1 + O(1/\sqrt{p})), & \text{if } \nu > 1. \end{cases}$$

Hence it follows that

$$S_q \ll s^{1/2} s_1^{1/2} 4^{\omega(s)} (2 + \varepsilon)^{\omega(q)},$$

in the notation of (1.5). Inserting this into our expression for  $\mathcal{S}_1^{(1)}$  yields

$$\mathcal{S}_1^{(1)} \ll s^{1/2} s_1^{1/2} q^{1/2} (2 + \varepsilon)^{\omega(b) + \omega(q)} 4^{\omega(s)} (\log P)^{2+\varepsilon} \left( \frac{b^{1/2} Y_0}{X} + \frac{a^{1/2}}{Y^{1/2}} \right). \quad (7.4)$$

Combining this with (7.3) we deduce that

$$\begin{aligned} \mathcal{S}_1 \ll & \frac{(2+\varepsilon)^{\omega(b)}(3+\varepsilon)^{\omega(q)}(\log P)^{2+\varepsilon}}{b^{1/4}q^{1/2}} \left( \frac{a^{1/2}XY}{Y_0} + \frac{a^{1/4}b^{1/4}X^{1/2}Y}{Y_0^{1/4}} \right) \\ & + s^{1/2}s_1^{1/2}q^{1/2}(2+\varepsilon)^{\omega(b)+\omega(q)}4^{\omega(s)}(\log P)^{2+\varepsilon} \left( \frac{b^{1/2}Y_0}{X} + \frac{a^{1/2}}{Y^{1/2}} \right), \end{aligned}$$

with  $Y_0 = |y_0| + Y$  and  $s, s_1$  given by (1.5). It will be important in what follows to note that

$$\mathcal{S}_1 = \mathcal{S}_1^{(0)} + \mathcal{S}_1^{(1)}, \quad (7.5)$$

where  $\mathcal{S}_1^{(0)}$  satisfies (7.3) and corresponds to the contribution from  $h = 0$  and likewise  $\mathcal{S}_1^{(1)}$  satisfies (7.4) and corresponds to the contribution from  $h \neq 0$ .

Drawing together (5.1), (5.4), (5.5) and our estimate for  $\mathcal{S}_1$ , we are now ready to record our final estimate for  $\mathcal{S}$ .

**Theorem 7.1.** — *Let  $\varepsilon > 0$ , let  $X, Y \geq 2$  and let  $P = 2abqXY$ . Let  $\text{rad}(q), s, s_1$  be given by (1.5) and let  $Y_0 = |y_0| + Y$ . Then we have*

$$\sum_{\substack{x,y \\ ax^2+by^3 \equiv 0 \pmod q}}^* \Gamma\left(\frac{x-x_0}{X}\right) \Gamma\left(\frac{y-y_0}{Y}\right) = \frac{\varphi(q)XY}{q^2} + O(E_1 + E_2 + E_3),$$

where

$$\begin{aligned} E_1 &= \frac{(2+\varepsilon)^{\omega(q)} \text{rad}(q)^{1/2} X}{q}, \\ E_2 &= \frac{(2+\varepsilon)^{\omega(b)}(3+\varepsilon)^{\omega(q)}}{b^{1/4}q^{1/2}} \left( \frac{a^{1/2}XY}{Y_0} + \frac{a^{1/4}b^{1/4}X^{1/2}Y}{Y_0^{1/4}} \right) (\log P)^{2+\varepsilon}, \\ E_3 &= s^{1/2}s_1^{1/2}q^{1/2}(2+\varepsilon)^{\omega(b)+\omega(q)}4^{\omega(s)} \left( \frac{b^{1/2}Y_0}{X} + \frac{a^{1/2}}{Y^{1/2}} \right) (\log P)^{2+\varepsilon}. \end{aligned}$$

This result is uniform in everything involved. In the application we have in mind it will prove crucial to have the precise dependence on all of the parameters worked out. Often however it suffices to bound the arithmetic functions involving  $a, b, q$  crudely. Assuming without loss of generality that  $a, b \leq q$ , Theorem 1.2 is now a simple consequence of Theorem 7.1 with  $x_0 = y_0 = 0$  and  $X = \mathbf{X}, Y = \mathbf{Y}$ .

## 8. Asymptotic formula for $M(B, \mathbf{X}, \mathbf{Y}; a, b; q)$

Let  $M(B, \mathbf{X}, \mathbf{Y}; a, b; q)$  be defined in (1.4), for  $B \geq 2$  and  $\mathbf{X}, \mathbf{Y} \geq 1$  and non-zero integers  $a, b, q$  such that  $q > 0$ . Again we may assume without loss of generality that  $a, b > 0$  and  $(ab, q) = 1$ . It is now time to turn to an asymptotic formula for  $M(B, \mathbf{X}, \mathbf{Y}; a, b; q)$ , building on the proof of Theorem 7.1. We will proceed under the hypothesis that

$$\log(2abq\mathbf{X}\mathbf{Y}) \ll \log B \quad (8.1)$$

and

$$a\mathbf{X}^2 \geq qB, \quad b\mathbf{Y}^3 \geq qB. \quad (8.2)$$

It will be convenient to set

$$\sigma := \left( \frac{qB}{a} \right)^{1/2}, \quad \tau := \left( \frac{qB}{b} \right)^{1/3}, \quad (8.3)$$

and to make a change of variables

$$s := \frac{x}{\sigma} \quad \text{and} \quad t := \frac{y}{\tau}. \quad (8.4)$$

Then the constraints  $0 < x \leq \mathbf{X}$ ,  $|y| \leq \mathbf{Y}$  and  $|ax^2 + by^3| \leq qB$  on  $x$  and  $y$  translate into the conditions

$$0 < s \leq \frac{\mathbf{X}}{\sigma}, \quad |t| \leq \frac{\mathbf{Y}}{\tau}, \quad |s^2 + t^3| \leq 1.$$

Likewise the assumption (8.2) becomes

$$\frac{\mathbf{X}}{\sigma} \geq 1, \quad \frac{\mathbf{Y}}{\tau} \geq 1. \quad (8.5)$$

For any region  $\Omega \subset \mathbb{R}^2$  set

$$\mathcal{S}(\Omega; a, b; q) := \# \{ (x, y) \in \mathbb{Z}^2 : (xy, q) = 1, ax^2 + by^3 \equiv 0 \pmod{q}, (s, t) \in \Omega \}. \quad (8.6)$$

Then

$$M(B, \mathbf{X}, \mathbf{Y}; a, b; q) = \mathcal{S}(\mathcal{R}; a, b; q), \quad (8.7)$$

with

$$\mathcal{R} = \left\{ (s, t) \in \mathbb{R}^2 : 0 < s \leq \frac{\mathbf{X}}{\sigma}, \quad |t| \leq \frac{\mathbf{Y}}{\tau}, \quad |s^2 + t^3| \leq 1 \right\}. \quad (8.8)$$

We split the region  $\mathcal{R}$  into two subregions

$$\mathcal{R}_1 := \left\{ (s, t) \in \mathcal{R} : 0 < s \leq L^{1/2} \right\}$$

and

$$\mathcal{R}_2 := \left\{ (s, t) \in \mathcal{R} : L^{1/2} < s \leq \frac{\mathbf{X}}{\sigma} \right\},$$

where throughout this paper, we set

$$L := \log \log (2000B).$$

Since  $B \geq 1$  it follows that  $L > 2$ .

Next, we want to bound the areas of regions  $\mathcal{R}_1$  and  $\mathcal{R}_2$ . To this end, we make the following observation. Let us assume that  $s > 2$  and  $(s, t) \in \mathcal{R}$  (in particular, this is satisfied if  $(s, t) \in \mathcal{R}_2$ ). Then we have  $t < 0$ . Indeed, if one assumes  $s > 2$  and  $t > 0$ , we would have  $2^2 < s^2 + t^3 \leq 1$ , which is impossible. It follows that

$$\begin{aligned} 1 \geq |s^2 + t^3| &= |s^2 - |t|^3| = \left| \left( |t| - s^{2/3} \right) \left( |t|^2 + |t|s^{2/3} + s^{4/3} \right) \right| \\ &> \left| |t| - s^{2/3} \right| s^{4/3} \\ &= \left| t + s^{2/3} \right| s^{4/3}. \end{aligned} \quad (8.9)$$

In this way we see that  $\text{meas}(\mathcal{R}_2) \ll L^{-1/6}$ . Moreover, using (8.5), we have

$$1 \ll \text{meas}(\mathcal{R}_1) \ll 1. \quad (8.10)$$

Hence

$$\text{meas}(\mathcal{R}_1) = \text{meas}(\mathcal{R}) \left( 1 + O \left( L^{-1/6} \right) \right). \quad (8.11)$$

Furthermore, we note that

$$0 < s \leq L^{1/2}, \quad |t| \ll L^{1/3}$$

for any  $(s, t) \in \mathcal{R}_1$ .

By our decomposition of  $\mathcal{R}$ , we have

$$\mathcal{S}(\mathcal{R}; a, b; q) = \mathcal{S}(\mathcal{R}_1; a, b; q) + \mathcal{S}(\mathcal{R}_2; a, b; q). \quad (8.12)$$

In what follows we describe how to bound the terms  $\mathcal{S}(\mathcal{R}_j; a, b; q)$  using sums involving smooth weights. To this end, for any smooth function  $\Psi : \mathbb{R} \rightarrow \mathbb{R}$  of rapid decay, we define

$$\mathcal{S}(\Psi; s_0, t_0, \mathcal{X}, \mathcal{Y}; a, b; q) := \sum_{\substack{x, y \in \mathbb{Z} \\ (xy, q) = 1 \\ ax^2 + by^3 \equiv 0 \pmod{q}}} \Psi\left(\frac{s - s_0}{\mathcal{X}}\right) \Psi\left(\frac{t - t_0}{\mathcal{Y}}\right), \quad (8.13)$$

where  $s, t$  depend on  $x, y$  via (8.4). We will choose smooth weights involving the Gaussian weight  $\Gamma$  so that we can apply our work above.

**8.1. Treatment of region  $\mathcal{R}_1$ .** — In what follows we set

$$\Delta := \frac{1}{L} = \frac{1}{\log \log(2000B)}.$$

A little thought reveals that region  $\mathcal{R}_1$  can be approximated by regions  $\mathcal{R}_1^-$  and  $\mathcal{R}_1^+$  that each are disjoint unions of  $O(L^4)$  half-open squares  $\mathcal{Q} = [s, s + \Delta) \times [t, t + \Delta)$  of side length  $\Delta$  such that

$$\mathcal{R}_1^- \subset \mathcal{R}_1 \subset \mathcal{R}_1^+$$

and

$$\left(1 - O(\Delta^{1/2})\right) \text{meas}(\mathcal{R}_1) < \text{meas}(\mathcal{R}_1^-) < \text{meas}(\mathcal{R}_1) < \text{meas}(\mathcal{R}_1^+) < \left(1 + O(\Delta^{1/2})\right) \text{meas}(\mathcal{R}_1).$$

Hence we have

$$\mathcal{S}(\mathcal{R}_1^-; a, b; q) \leq \mathcal{S}(\mathcal{R}_1; a, b; q) \leq \mathcal{S}(\mathcal{R}_1^+; a, b; q).$$

Let  $\mathcal{M}^\pm$  be the set of squares whose union is  $\mathcal{R}_1^\pm$ . It follows that

$$\mathcal{S}(\mathcal{R}_1^\pm; a, b; q) = \sum_{\mathcal{Q} \in \mathcal{M}^\pm} \mathcal{S}(\mathcal{Q}; a, b; q). \quad (8.14)$$

Next let  $\chi$  be the characteristic function of the interval  $[-1/2, 1/2)$ . We approximate  $\chi$  by the smooth weight functions  $\Phi_+, \Phi_- : \mathbb{R} \rightarrow \mathbb{R}$  constructed in Lemma 3.1. There it is proved that  $\Phi_+, \Phi_-$  have rapid decay and satisfy the inequalities  $\Phi_-(x) \leq \chi(x) \leq \Phi_+(x)$  for all  $x \in \mathbb{R}$ . From the definitions (8.6) and (8.13) and we deduce that

$$\mathcal{S}(\Phi_-; s_1, t_1, \Delta, \Delta; a, b; q) \leq \mathcal{S}(\mathcal{Q}; a, b; q) \leq \mathcal{S}(\Phi_+; s_1, t_1, \Delta, \Delta; a, b; q), \quad (8.15)$$

where  $(s_1, t_1)$  is the centre of the square  $\mathcal{Q}$ . Using (8.13) and the definitions of  $\Phi_-$  and  $\Phi_+$  in §3.1, we further have

$$\begin{aligned} \mathcal{S}(\Phi_\pm; s_1, t_1, \Delta, \Delta; a, b; q) &= L^2 \int_{-1/2 \mp \Delta^{1/2}}^{1/2 \pm \Delta^{1/2}} \int_{-1/2 \mp \Delta^{1/2}}^{1/2 \pm \Delta^{1/2}} \mathcal{S}(\Gamma; s_1 + \Delta\mu, t_1 + \Delta\nu, \Delta^2, \Delta^2; a, b; q) d\mu d\nu \\ &\quad \pm \varepsilon_1 L \int_{-1/2 \mp \Delta^{1/2}}^{1/2 \pm \Delta^{1/2}} \mathcal{S}(\Gamma; s_1 + \Delta\mu, t_1, \Delta^2, \Delta; a, b; q) d\mu \\ &\quad \pm \varepsilon_1 L \int_{-1/2 \mp \Delta^{1/2}}^{1/2 \pm \Delta^{1/2}} \mathcal{S}(\Gamma; s_1, t_1 + \Delta\nu, \Delta, \Delta^2; a, b; q) d\nu \\ &\quad + \varepsilon_1^2 \mathcal{S}(\Gamma; s_1, t_1, \Delta, \Delta; a, b; q), \end{aligned} \quad (8.16)$$

where

$$\varepsilon_1 = 24 \exp(-L) = \frac{24}{\log(2000B)}.$$

In this way, the estimation of  $\mathcal{S}(\mathcal{R}_1; a, b; q)$  has been reduced to the task of estimating sums of the form  $\mathcal{S}(\Gamma; s_0, t_0, \mathcal{X}, \mathcal{Y}; a, b; q)$  in (8.13), which are precisely the sums considered in Theorem 7.1, with

$$X := \sigma \mathcal{X}, \quad x_0 := \sigma s_0, \quad Y := \tau \mathcal{Y}, \quad y_0 := \tau t_0. \quad (8.17)$$

We then have

$$s_0 = s_1 + (i-1)\Delta\mu, \quad t_0 = t_1 + (j-1)\Delta\nu, \quad \mathcal{X} = \Delta^i, \quad \mathcal{Y} = \Delta^j,$$

for  $i, j \in \{1, 2\}$ . Since  $t_1 \ll L$ ,  $\nu \ll 1$  and  $y_0 = t_0 Y / \mathcal{Y}$  we may deduce that

$$Y \leq Y + |y_0| = Y_0 \ll YL^3 = Y(\log \log 2000B)^3.$$

Using this and the fact that  $(\log \log 2000B)^{-2} = \Delta^2 \leq \mathcal{X}, \mathcal{Y} \leq \Delta = (\log \log 2000B)^{-1}$ , together with (8.3) and (8.17), we deduce from Theorem 7.1 that

$$\begin{aligned} \mathcal{S}(\Gamma; s_0, t_0, \mathcal{X}, \mathcal{Y}; a, b; q) &= \frac{\varphi(q)}{q} \cdot \frac{B^{5/6}}{a^{1/2}b^{1/3}q^{1/6}} \cdot \mathcal{X}\mathcal{Y} + O\left(\frac{(2+\varepsilon)^{\omega(q)}B^{1/2}}{a^{1/2}}\right) \\ &\quad + O\left(\frac{(2+\varepsilon)^{\omega(b)}(3+\varepsilon)^{\omega(q)}}{b^{1/4}} \cdot B^{1/2}(\log B)^{2+\varepsilon}\right) \\ &\quad + O\left((ss_1)^{1/2}(2+\varepsilon)^{\omega(b)}(2+\varepsilon)^{\omega(q)}4^{\omega(s)} \cdot \frac{a^{1/2}b^{1/6}q^{1/3}(\log B)^{2+\varepsilon}}{B^{1/6}}\right), \end{aligned}$$

where we note that  $\text{rad}(q) \leq q$ . Let

$$f_\varepsilon(a, b; q) := \frac{(2+\varepsilon)^{\omega(q)}}{a^{1/2}} + \frac{(3+\varepsilon)^{\omega(q)}}{b^{1/4-\varepsilon}}. \quad (8.18)$$

Then on inserting this into (8.14), (8.15) and (8.16), and recalling Lemma 3.1 and (8.11), we conclude that

$$\begin{aligned} \mathcal{S}(\mathcal{R}_1; a, b; q) &= \frac{\varphi(q)}{q} \cdot \frac{B^{5/6}}{a^{1/2}b^{1/3}q^{1/6}} \cdot \text{meas}(\mathcal{R}) \left(1 + O\left(L^{-1/6}\right)\right) \\ &\quad + O\left(f_\varepsilon(a, b; q)B^{1/2}(\log B)^{2+\varepsilon}\right) \\ &\quad + O\left((ss_1)^{1/2}(2+\varepsilon)^{\omega(b)+\omega(q)}4^{\omega(s)} \cdot \frac{a^{1/2}b^{1/6}q^{1/3}(\log B)^{2+\varepsilon}}{B^{1/6}}\right). \end{aligned} \quad (8.19)$$

**8.2. Treatment of region  $\mathcal{R}_2$ .** — Next we deal with region  $\mathcal{R}_2$ . Heuristically,  $\mathcal{R}_2$  should yield an error contribution because the area of  $\mathcal{R}_2$  is small compared to that of  $\mathcal{R}_1$ . Therefore it will suffice to cover  $\mathcal{R}_2$  by a larger region whose area is still small compared to that of  $\mathcal{R}_1$ .

Using (8.9), we may cover the region  $\mathcal{R}_2$  by a union of  $O(\log B)$  regions of the form

$$\mathcal{R}(U) := \left\{ (s, t) \in \mathbb{R}^2 : \frac{4}{3}U < s \leq \frac{5}{3}U, \quad t < 0, \quad \left|t + s^{2/3}\right| < \frac{1}{U^{4/3}} \right\}, \quad (8.20)$$

where

$$\frac{3}{4}L^{1/2} \leq U \leq \frac{\mathbf{X}}{\sigma}.$$

We observe that in certain cases the condition  $U \leq \mathbf{X}/\sigma$  above can be strengthened. Since  $s \asymp |t|^{3/2}$  and  $|t| \leq \mathbf{Y}/\tau$  in region  $\mathcal{R}_2$ , it follows that  $s \ll (\mathbf{Y}/\tau)^{3/2}$ , and hence it suffices to suppose that

$$\frac{3}{4}L^{1/2} \leq U \ll \mathbf{U}, \quad (8.21)$$

where

$$U := \min \left\{ \frac{\mathbf{X}}{\sigma}, \left( \frac{\mathbf{Y}}{\tau} \right)^{3/2} \right\}. \quad (8.22)$$

In particular  $U \geq 1$  by (8.2).

We may further cover  $\mathcal{R}(U)$  by a union of (small) rectangles

$$\mathcal{Q}(s_0) := [s_0 - \mathcal{X}, s_0 + \mathcal{X}] \times [t_0 - \mathcal{Y}, t_0 + \mathcal{Y}]$$

of side lengths  $2\mathcal{X}$  and  $2\mathcal{Y}$ , centred at points  $(s_0, t_0)$  with

$$\frac{4}{3}U \leq s_0 \leq \frac{5}{3}U \quad \text{and} \quad t_0 = -s_0^{2/3},$$

and satisfying the condition that

$$\{(s, t) \in \mathcal{R}(U) : s_0 - \mathcal{X} \leq s \leq s_0 + \mathcal{X}\} \subseteq \mathcal{Q}(s_0). \quad (8.23)$$

Let  $T$  be a parameter with

$$U^{-1} \leq T \leq U^{1-\varepsilon}, \quad (8.24)$$

to be chosen later appropriately. We set

$$\mathcal{X} := T, \quad \mathcal{Y} := 2TU^{-1/3}. \quad (8.25)$$

In the following, we suppose that  $s_0 \in [4U/3, 5U/3]$  and show that the condition (8.23) then holds automatically under the above choices of  $\mathcal{X}$  and  $\mathcal{Y}$ .

Considering the definition of  $\mathcal{R}(U)$ , and taking into account that the function  $f(s) = -s^{2/3}$  is monotonically decreasing for  $s > 0$ , we have

$$\begin{aligned} & \{(s, t) \in \mathcal{R}(U) : s_0 - \mathcal{X} \leq s \leq s_0 + \mathcal{X}\} \\ & \subseteq [s_0 - \mathcal{X}, s_0 + \mathcal{X}] \times \left[ -(s_0 + \mathcal{X})^{2/3} - \frac{1}{U^{4/3}}, -(s_0 - \mathcal{X})^{2/3} + \frac{1}{U^{4/3}} \right], \end{aligned}$$

and hence it suffices to prove that

$$t_0 - \mathcal{Y} \leq -(s_0 + \mathcal{X})^{2/3} - \frac{1}{U^{4/3}} \quad (8.26)$$

and

$$t_0 + \mathcal{Y} \geq -(s_0 - \mathcal{X})^{2/3} + \frac{1}{U^{4/3}}. \quad (8.27)$$

Obviously we have

$$-(s_0 + \mathcal{X})^{2/3} = -s_0^{2/3} \left( 1 + \frac{\mathcal{X}}{s_0} \right)^{2/3} > -s_0^{2/3} \left( 1 + \frac{\mathcal{X}}{s_0} \right) \geq -s_0^{2/3} - \frac{T}{U^{1/3}} = t_0 - \frac{T}{U^{1/3}}.$$

Similarly,

$$-(s_0 - \mathcal{X})^{2/3} = -s_0^{2/3} \left( 1 - \frac{\mathcal{X}}{s_0} \right)^{2/3} \leq -s_0^{2/3} \left( 1 - \frac{\mathcal{X}}{s_0} \right) \leq -s_0^{2/3} + \frac{T}{U^{1/3}} = t_0 + \frac{T}{U^{1/3}}.$$

Recalling that  $\mathcal{Y} = 2TU^{-1/3}$  and  $T \geq U^{-1}$ , we therefore obtain (8.26) and (8.27), and so (8.23) holds.

Now instead of covering  $\mathcal{R}(U)$  discretely by rectangles  $\mathcal{Q}(s_0)$  and summing up their contributions, we rather integrate, which will be useful because we shall later take advantage of cancellations occurring in this integration. To be precise, we claim that

$$\mathcal{S}(\mathcal{R}(U); a, b, q) \leq \frac{1}{T} \int_{-\infty}^{\infty} W\left(\frac{s_0}{U}\right) \mathcal{S}(\mathcal{Q}(s_0); a, b, q) ds_0, \quad (8.28)$$

where  $W$  is a smooth weight function with

$$\text{supp}(W) \subseteq [1, 2] \quad \text{and} \quad W(u) = 1 \text{ for } \frac{4}{3} \leq u \leq \frac{5}{3}. \quad (8.29)$$

The inequality (8.28) is seen as follows. Using (8.6), we have

$$\begin{aligned} & \frac{1}{T} \int_{-\infty}^{\infty} W\left(\frac{s_0}{U}\right) \mathcal{S}(\mathcal{Q}(s_0); a, b; q) ds_0 \\ &= \frac{1}{T} \int_{-\infty}^{\infty} W\left(\frac{s_0}{U}\right) \# \left\{ (x, y) \in \mathbb{Z}^2 : \begin{array}{l} (xy, q) = 1, \quad ax^2 + by^3 \equiv 0 \pmod{q}, \\ (s, t) \in \mathcal{Q}(s_0) \end{array} \right\} ds_0 \\ &\geq \sum_{x, y} \frac{1}{T} \int_{\mathcal{M}(s, t)} W\left(\frac{s_0}{U}\right) ds_0, \end{aligned}$$

where the sum is over  $(x, y) \in \mathbb{Z}^2$  such that  $ax^2 + by^3 \equiv 0 \pmod{q}$ ,  $(xy, q) = 1$  and  $(s, t) \in \mathcal{R}(U)$ , and where

$$\mathcal{M}(s, t) := \left\{ s_0 \in \left[ \frac{4}{3}U, \frac{5}{3}U \right] : (s, t) \in \mathcal{Q}(s_0) \right\}.$$

By (8.23) and the fact that  $\mathcal{X} = T$ , we have

$$\mathcal{M}(s, t) = [4U/3, 5U/3] \cap [s - T, s + T]$$

for any  $(s, t) \in \mathcal{R}(U)$ . Moreover, by (8.29), we have  $W(s_0/U) = 1$  whenever  $s_0 \in \mathcal{M}(s, t)$ . Hence it follows that

$$\sum_{x, y} \frac{1}{T} \int_{\mathcal{M}(s, t)} W\left(\frac{s_0}{U}\right) ds_0 = \sum_{x, y} \frac{\text{meas}(\mathcal{M}(s, t))}{T} \geq \sum_{x, y} 1 = \mathcal{S}(\mathcal{R}(U); a, b; q).$$

Combining these inequalities we therefore obtain the desired inequality (8.28).

From (8.13) and (8.28), we further deduce that

$$\mathcal{S}(\mathcal{R}(U); a, b; q) \ll \frac{1}{T} \int_{-\infty}^{\infty} W\left(\frac{s_0}{U}\right) \mathcal{S}(\Gamma; s_0, t_0, \mathcal{X}, \mathcal{Y}; a, b; q) ds_0, \quad (8.30)$$

where  $\Gamma$  is defined as in (3.1) and  $\mathcal{S}(\Gamma; s_0, t_0, \mathcal{X}, \mathcal{Y}; a, b; q)$  is given in (8.13). Thus the route is open for a modification of the approach used to prove Theorem 7.1. Using (5.2) and (8.30), we obtain

$$\mathcal{S}(\mathcal{R}(U); a, b; q) \ll \mathcal{S}(U) := \frac{XY}{q^2 T} \sum_{m \in \mathbb{Z}} \sum_{n \in \mathbb{Z}} \Gamma\left(\frac{mX}{q}\right) \Gamma\left(\frac{nY}{q}\right) \mathcal{J}(m, n; q) \mathcal{E}(m, n; q), \quad (8.31)$$

where  $X, Y$  are given by (8.3) and (8.17), and furthermore,

$$\mathcal{J}(m, n; q) := \int_{-\infty}^{\infty} W\left(\frac{s_0}{U}\right) e\left(-\frac{m\sigma s_0 - n\tau s_0^{2/3}}{q}\right) ds_0.$$

As previously we split the sum on the right-hand side of (8.31) into

$$\mathcal{S}(U) = \mathcal{S}_0(U) + \mathcal{S}_1(U) + \mathcal{S}_2(U), \quad (8.32)$$

where  $\mathcal{S}_0(U)$  denotes the contribution of  $m = 0$  and  $n = 0$ ,  $\mathcal{S}_1(U)$  the contribution of  $m \neq 0$  and  $n$  arbitrary, and  $\mathcal{S}_2(U)$  the contribution of  $m = 0$  and  $n \neq 0$ . If  $m = 0$  and  $n = 0$ , then we note that

$$\mathcal{J}(0, 0; q) = O(U).$$

Since  $\mathcal{E}(0, 0; q) = \varphi(q)$  we obtain

$$\mathcal{S}_0(U) \ll \frac{\varphi(q)}{q^2} \cdot XY \cdot \frac{U}{T}. \quad (8.33)$$

Next if  $m = 0$  and  $n \neq 0$ , then using the remark after Lemma 2.1 in [17], we get

$$\mathcal{J}(0, n; q) = O\left(\frac{U^{1/3}q}{|n|\tau}\right).$$

Hence

$$\mathcal{S}_2(U) \ll \frac{XYU^{1/3}}{\tau q T} \sum_{n \neq 0} \frac{1}{|n|} \cdot \Gamma\left(\frac{nY}{q}\right) |\mathcal{E}(0, n; q)|.$$

Arguing as in the proof of (5.5) we easily obtain

$$\mathcal{S}_2(U) \ll \frac{XY \operatorname{rad}(q)^{1/2} U^{1/3}}{\tau q T} \cdot (\log B)(2 + \varepsilon)^{\omega(q)}. \quad (8.34)$$

It remains to estimate

$$\mathcal{S}_1(U) = \frac{XY}{q^2 T} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \Gamma\left(\frac{mX}{q}\right) \Gamma\left(\frac{nY}{q}\right) \mathcal{J}(m, n; q) \mathcal{E}(m, n; q).$$

Recall from (8.17) that  $x_0 = \sigma s_0$ . Pulling out the integral and making a change of variables  $s_0 \rightarrow x_0$ , we obtain

$$\mathcal{S}_1(U) = \frac{XY}{\sigma q^2 T} \int_{-\infty}^{\infty} \sum_{m \neq 0} \sum_{n \in \mathbb{Z}} \Gamma\left(\frac{mX}{q}\right) \Gamma\left(\frac{nY}{q}\right) W\left(\frac{x_0}{\sigma U}\right) e\left(-\frac{mx_0 + ny_0}{q}\right) \mathcal{E}(m, n; q) dx_0,$$

where

$$y_0 = -\delta x_0^{2/3} \quad (8.35)$$

and

$$\delta := \frac{\tau}{\sigma^{2/3}} = \left(\frac{a}{b}\right)^{1/3}, \quad (8.36)$$

by (8.3).

Denote the term  $\mathcal{S}_1$  in (5.6) by  $\mathcal{S}_1(x_0, y_0)$ . Then

$$\mathcal{S}_1(U) = \frac{1}{\sigma T} \int_{-\infty}^{\infty} W\left(\frac{x_0}{\sigma U}\right) \mathcal{S}_1(x_0, y_0) dx_0.$$

Before we estimate  $\mathcal{S}_1(U)$ , we recall the main steps in the estimation of  $\mathcal{S}_1 = \mathcal{S}_1(x_0, y_0)$ . Our starting point was the identity (5.21). In (6.1) the sum  $S(u)$  therein was further split into sums  $S(u, t)$  for which we derived the identity (6.6). The exponential sum  $F_x(h)$  is independent of  $x_0$  and  $y_0$  and was estimated separately for the cases  $h \neq 0$  and  $h = 0$ . In the case  $h \neq 0$  we got an estimate in which only one factor depends on  $h$ , namely the factor  $(h, b_1)^{1/2}$ . The function  $\hat{\Psi}$  in (6.6) can be written as an exponential integral and depends on  $y_0$ . Using (6.25) its estimation was the subject of §3.2. Furthermore, the series

$$\sum_{h \neq 0} |\hat{\Psi}(h\alpha)|(h, b_1)^{1/2}$$

was estimated in (6.29). For  $\hat{\Psi}(0)$  we obtained the estimate (6.32). Putting everything together, we arrived at an estimate for  $\mathcal{S}_1$  which was recorded in (7.3), (7.4) and (7.5). We recall that  $\mathcal{S}_1^{(0)}$  and  $\mathcal{S}_1^{(1)}$  are the contributions of  $h = 0$  and  $h \neq 0$ , respectively. Hence the terms on the right-hand sides of (6.29) and (6.32) correspond to the terms on the right-hand sides of (7.4) and (7.3), respectively.



Our strategy for the estimation of  $\mathcal{S}_1(U)$  is as follows. We denote the terms  $\mathcal{S}_1^{(i)}$  in (7.5) by  $\mathcal{S}_1^{(i)}(x_0, y_0)$ , for  $i = 0, 1$ . Then

$$\mathcal{S}_1(U) = \mathcal{S}_1^{(0)}(U) + \mathcal{S}_1^{(1)}(U), \quad (8.37)$$

where, for  $i = 0, 1$ ,

$$\mathcal{S}_1^{(i)}(U) := \frac{1}{\sigma T} \int_{-\infty}^{\infty} W\left(\frac{x_0}{\sigma U}\right) \mathcal{S}_1^{(i)}(x_0, y_0) dx_0.$$

We estimate  $\mathcal{S}_1^{(0)}(U)$  by integrating trivially the right-hand side of (7.3), giving

$$\mathcal{S}_1^{(0)}(U) \ll \frac{(\log B)^{2+\varepsilon}}{\sigma T} \int_{\sigma U}^{2\sigma U} \frac{(2+\varepsilon)^{\omega(b)}(3+\varepsilon)^{\omega(q)}}{b^{1/4}q^{1/2}} \left( \frac{a^{1/2}XY}{Y_0} + \frac{a^{1/4}b^{1/4}X^{1/2}Y}{Y_0^{1/4}} \right) dx_0.$$

Taking  $Y_0 = |y_0| + Y > |y_0| = \delta x_0^{2/3}$  by (8.35) into account, we see that

$$\int_{\sigma U}^{2\sigma U} Y_0^{-\theta} dx_0 < \frac{1}{\delta^\theta} \int_{\sigma U}^{2\sigma U} x_0^{-2\theta/3} dx_0 \ll \frac{(\sigma U)^{1-2\theta/3}}{\delta^\theta},$$

for any  $\theta \in [0, 3/2)$ . Using (8.36) we may therefore conclude that

$$\mathcal{S}_1^{(0)}(U) \ll \frac{(2+\varepsilon)^{\omega(b)}(3+\varepsilon)^{\omega(q)}(\log B)^{2+\varepsilon}}{b^{1/4}q^{1/2}T} \left( \frac{a^{1/2}XYU^{1/3}}{\tau} + \frac{a^{1/4}b^{1/4}X^{1/2}YU^{5/6}}{\tau^{1/4}} \right). \quad (8.38)$$

**8.3. Estimation of  $\mathcal{S}_1^{(1)}(U)$ .** — We now turn to the more delicate estimation of  $\mathcal{S}_1^{(1)}(U)$ . For this we will first write  $\mathcal{S}_1^{(1)}$  explicitly as an identity. Then we pull in the integral and integrate non-trivially. Roughly speaking, we will proceed to compare termwise the sizes of the integrals and integrands. This will tell us how the estimate (7.4) needs to be adjusted to obtain a final estimate for  $\mathcal{S}_1^{(1)}(U)$ .

Recalling (5.21), (6.1) and (6.6), we may write

$$\begin{aligned} \mathcal{S}_1^{(1)} &= \frac{XY}{q^2} \sum_{\substack{(6r_1, r_2)=1 \\ r_1 r_2 = q}} \sum_{\substack{d_1 | r_1, d_2 | r_2 \\ \text{rad}(\hat{r}_1)^2 | e_1 \\ \mu^2(e_2)=1}} \sum'_{d'_1 | r_1} d'_2 \sqrt{r_1 d_1} \sum_{\substack{u \neq 0 \\ (u, e_1 e_2)=1}} \Gamma\left(\frac{2^{f_2} d_1 d_2 u X}{q}\right) \\ &\times \sum_{\substack{t | e'_1 \\ (t, e)=1}} \mu(t) \left(\frac{t}{e}\right) e\left(-\frac{2^{f_2} d_1 d_2 u x_0}{q}\right) \frac{\alpha}{D} \sum_{x=1}^D \sum_{y=1}^D e\left(\frac{xy}{D}\right) Q''(y) \sum_{h \neq 0} \hat{\Psi}(\alpha h) F_x(h), \end{aligned}$$

where  $\alpha$  is defined as in (6.24). It follows that

$$\begin{aligned} \mathcal{S}_1^{(1)}(U) &\ll \frac{XY}{q^2} \sum_{\substack{(6r_1, r_2)=1 \\ r_1 r_2 = q}} \sum_{\substack{d_1 | r_1, d_2 | r_2 \\ \text{rad}(\hat{r}_1)^2 | e_1 \\ \mu^2(e_2)=1}} \sum'_{d'_1 | r_1} d'_2 \sqrt{r_1 d_1} \sum_{\substack{u \neq 0 \\ (u, e_1 e_2)=1}} \Gamma\left(\frac{2^{f_2} d_1 d_2 u X}{q}\right) \\ &\times \sum_{\substack{t | e'_1 \\ (t, e)=1}} |\mu(t)| \alpha \sum_{x=1}^D \sum_{h \neq 0} |F_x(h)| \\ &\times \frac{1}{\sigma T} \left| \int_{-\infty}^{\infty} W\left(\frac{x_0}{\sigma U}\right) e\left(-\frac{2^{f_2} d_1 d_2 u x_0}{q}\right) \hat{\Psi}_{y_0}(\alpha h) dx_0 \right|, \end{aligned}$$

where the subscript  $y_0$  indicates the dependency of the function  $\hat{\Psi}$  on  $y_0$ , which in turn depends on  $x_0$ . A key point in the estimation of  $\mathcal{S}_1^{(1)}$  was our bound (6.29) for

$$\sum_{h \neq 0} \left| \hat{\Psi}_{y_0}(\alpha h) \right| (h, b_1)^{1/2}.$$

We recall that the factor  $(h, b_1)^{1/2}$  that appears here came from our estimate of  $F_x(h)$  in §6. For the estimation of  $\mathcal{S}_1^{(1)}(U)$ , we require a bound for

$$\mathcal{L} := \frac{1}{\sigma T} \sum_{h \neq 0} \left| \int_{-\infty}^{\infty} W\left(\frac{x_0}{\sigma U}\right) e\left(-\frac{2^{f_2} d_1 d_2 u x_0}{q}\right) \hat{\Psi}_{y_0}(\alpha h) dx_0 \right| (h, b_1)^{1/2} \quad (8.39)$$

instead. After deriving a bound for this sum, we shall compare the terms appearing in it with the corresponding terms on the right-hand side of (6.29). This will allow us to infer a bound for  $\mathcal{S}_1^{(1)}(U)$  directly from our bound (7.4) for  $\mathcal{S}_1^{(1)}$ .

We recall that

$$\hat{\Psi}_{y_0}(\alpha h) = F(\gamma + \alpha h, \beta) = F\left(\frac{y_0}{Y} + \alpha h, \beta\right) = F\left(-\frac{\delta x_0^{2/3}}{Y} + \alpha h, \beta\right),$$

where  $\alpha$  and  $\beta$  are defined in (6.24) and  $\delta$  is defined as in (8.36). Hence the integrand in (8.39) is

$$\mathcal{J}_h(x_0) := W\left(\frac{x_0}{\sigma U}\right) e\left(-\frac{2^{f_2} d_1 d_2 u x_0}{q}\right) F\left(-\frac{\delta x_0^{2/3}}{Y} + \alpha h, \beta\right).$$

As in (6.28) the contribution of  $|h| > B^\kappa$  to (8.39) is negligible if  $\kappa$  is large enough. We split the remaining contribution to (8.39) as follows.

$$\sum_{1 \leq |h| \leq B^\kappa} \left| \int_{-\infty}^{\infty} \mathcal{J}_h(x_0) dx_0 \right| (h, b_1)^{1/2} = \sum_{i=1,2} \sum_{1 \leq |h| \leq B^\kappa} \left| \int_{\mathcal{M}_i} \mathcal{J}_h(x_0) dx_0 \right| (h, b_1)^{1/2}, \quad (8.40)$$

where

$$\mathcal{M}_1 := \left\{ x_0 \in [\sigma U, 2\sigma U] : \left| -\frac{\delta x_0^{2/3}}{Y} + \alpha h \right| \leq 1 \right\} \quad \text{and} \quad \mathcal{M}_2 := [\sigma U, 2\sigma U] \setminus \mathcal{M}_1.$$

Let us begin with the treatment of the integral over  $\mathcal{M}_1$ . Using (3.4) we have

$$\mathcal{J}_h(x_0) \ll 1. \quad (8.41)$$

It remains to estimate the Lebesgue measure of  $\mathcal{M}_1$ . To this end, we note that (8.17), (8.24), (8.25) and (8.36) imply

$$\frac{\delta x_0^{2/3}}{Y} \asymp \frac{U}{T} \gg U^\varepsilon \quad \text{if } x_0 \in [\sigma U, 2\sigma U]. \quad (8.42)$$

Hence we have

$$\alpha |h| \asymp \frac{\delta x_0^{2/3}}{Y} \quad \text{if } x_0 \in [\sigma U, 2\sigma U] \quad \text{and} \quad \left| -\frac{\delta x_0^{2/3}}{Y} + \alpha h \right| \leq 1. \quad (8.43)$$

Using this we claim that  $\mathcal{M}_1$  is an interval of length

$$\text{meas}(\mathcal{M}_1) \ll \frac{\sigma U}{\alpha |h|}. \quad (8.44)$$

To see this we let  $m_1$  and  $m_2$  be the two endpoints of the interval  $\mathcal{M}_1$  in question. We have  $m_1 \asymp m_2 \asymp \sigma U$ . Under this condition Taylor's theorem gives

$$m_1^{2/3} - m_2^{2/3} \asymp (m_1 - m_2) m_2^{-1/3}.$$

Furthermore, by the definition of our interval  $\mathcal{M}_1$ , we have

$$\frac{\delta m_1^{2/3}}{Y} - \frac{\delta m_2^{2/3}}{Y} \ll 1.$$

Combining these two inequalities, we deduce that

$$m_1 - m_2 \ll \frac{Y m_2^{1/3}}{\delta} \ll \frac{Y \sigma^{1/3} U^{1/3}}{\delta} \ll \frac{\sigma U}{\alpha |h|},$$

by (8.43) and  $x_0 \in [\sigma U, 2\sigma U]$ . This therefore gives (8.44). Employing (3.17), (8.41), (8.44) and partial summation, we get

$$\sum_{1 \leq |h| \leq B^\kappa} \left| \int_{\mathcal{M}_1} \mathcal{J}_h(x_0) dx_0 \right| (h, b_1)^{1/2} \ll \frac{\sigma U}{\alpha} \sum_{1 \leq |h| \leq B^\kappa} \frac{(h, b_1)^{1/2}}{|h|} \ll \frac{\sigma U \log B}{\alpha} \cdot (1 + \varepsilon)^{\omega(b_1)}. \quad (8.45)$$

Now we turn to the second integral on the right-hand side of (8.40). We write  $\mathcal{M}_2 = \mathcal{M}_2^- \cup \mathcal{M}_2^+$ , where

$$\begin{aligned} \mathcal{M}_2^- &:= \left\{ x_0 \in [\sigma U, 2\sigma U] : -\frac{\delta x_0^{2/3}}{Y} + \alpha h \leq -1 \right\} \\ \mathcal{M}_2^+ &:= \left\{ x_0 \in [\sigma U, 2\sigma U] : -\frac{\delta x_0^{2/3}}{Y} + \alpha h \geq 1 \right\}. \end{aligned}$$

If  $x_0 \in \mathcal{M}_2^-$ , then we have

$$\begin{aligned} F\left(-\frac{\delta x_0^{2/3}}{Y} + \alpha h, \beta\right) &= \frac{2^{1/2} \exp\left(-\pi \left| -\delta x_0^{2/3}/Y + \alpha h \right| / (3\beta)\right)}{\left(3 \left| -\delta x_0^{2/3}/Y + \alpha h \right| \beta\right)^{1/4}} \\ &\quad \times \cos\left(2\pi \left(\frac{1}{8} - \frac{2 \left| -\delta x_0^{2/3}/Y + \alpha h \right|^{3/2}}{3^{3/2} \beta^{1/2}}\right)\right) + O\left(\frac{1}{\left| -\delta x_0^{2/3}/Y + \alpha h \right|}\right) \end{aligned}$$

by (3.12). It follows that

$$\int_{\mathcal{M}_2^-} \mathcal{J}_h(x_0) dx_0 \ll \frac{1}{\beta^{1/4}} \cdot \left| \int_{\mathcal{M}_2^-} \frac{Z(x_0)}{\left| -\delta x_0^{2/3}/Y + \alpha h \right|^{1/4}} dx_0 \right| + \int_{\mathcal{M}_2^-} \frac{dx_0}{\left| -\delta x_0^{2/3}/Y + \alpha h \right|}, \quad (8.46)$$

where

$$\begin{aligned} Z(x_0) &:= W\left(\frac{x_0}{\sigma U}\right) \exp\left(-\frac{\pi \left| -\delta x_0^{2/3}/Y + \alpha h \right|}{3\beta}\right) e\left(-\frac{2^{f_2} d_1 d_2 u x_0}{q}\right) \\ &\quad \times \cos\left(2\pi \left(\frac{1}{8} - \frac{2 \left| -\delta x_0^{2/3}/Y + \alpha h \right|^{3/2}}{3^{3/2} \beta^{1/2}}\right)\right). \end{aligned} \quad (8.47)$$

We begin by estimating the second integral on the right-hand side of (8.46). First assume that  $\alpha h \geq \delta(\sigma U)^{2/3}/(2Y)$ . Let  $x_1$  be a positive real number such that

$$\alpha h = \frac{\delta x_1^{2/3}}{Y} - 1.$$

If  $x_1 > 2\sigma U$ , then  $\mathcal{M}_2^-$  is empty. Thus we may assume that  $\sigma U/3 \leq x_1 \leq 2\sigma U$  since  $\delta x_1^{2/3}/Y \geq \alpha h \geq \delta(\sigma U)^{2/3}/(2Y)$ . Hence we have

$$\int_{\mathcal{M}_2^-} \frac{dx_0}{\left| -\delta x_0^{2/3}/Y + \alpha h \right|} \leq \int_{x_1}^{2\sigma U} \frac{dx_0}{\left| -\delta x_0^{2/3}/Y + \alpha h \right|}.$$

Now making a change of variables  $z = \delta x_0^{2/3}/Y - \alpha h$ , we deduce that

$$\int_{\mathcal{M}_2^-} \frac{dx_0}{\left| -\delta x_0^{2/3}/Y + \alpha h \right|} \leq \frac{3}{2} \int_1^a \frac{Y x_0^{1/3}}{\delta z} dz,$$

where  $a = \delta(2\sigma U)^{2/3}/Y - \alpha h$ . Similarly as in (8.42) and (8.43), we have  $\delta(2\sigma U)^{2/3}/Y \asymp U/T$  and

$$\alpha|h| \asymp \frac{\delta x_1^{2/3}}{Y} \asymp \frac{\delta(\sigma U)^{2/3}}{Y}.$$

Hence  $\log a \ll \log U$ . Using this and the fact that  $x_0 \asymp \sigma U$ , it follows that

$$\int_{\mathcal{M}_2^-} \frac{dx_0}{\left| -\delta x_0^{2/3}/Y + \alpha h \right|} \ll \min \left\{ \frac{Y(\sigma U)^{1/3}}{\delta}, \frac{\sigma U}{\alpha|h|} \right\} \log U. \quad (8.48)$$

If  $\alpha h < \delta(\sigma U)^{2/3}/(2Y)$ , then  $\left| -\delta x_0^{2/3}/Y + \alpha h \right| \gg \delta(\sigma U)^{2/3}/Y$  and  $\left| -\delta x_0^{2/3}/Y + \alpha h \right| \gg \alpha|h|$  for every  $x_0 \in [\sigma U, 2\sigma U]$ , and hence (8.48) holds trivially.

Next, we estimate the first integral on the right-hand side of (8.46). Using integration by parts, we write this integral as

$$\begin{aligned} \int_{\mathcal{M}_2^-} \frac{Z(x_0)}{\left| -\delta x_0^{2/3}/Y + \alpha h \right|^{1/4}} dx_0 &= \frac{1}{(\delta(2\sigma U)^{2/3}/Y - \alpha h)^{1/4}} \int_{x_2}^{2\sigma U} Z(x_0) dx_0 \\ &\quad + \frac{\delta}{6Y} \int_{x_2}^{2\sigma U} \frac{1}{z^{1/3} (\delta z^{2/3}/Y - \alpha h)^{5/4}} \left( \int_{x_2}^z Z(x_0) dx_0 \right) dz, \end{aligned} \quad (8.49)$$

where  $x_2$  is the lower endpoint of the interval  $\mathcal{M}_2^-$  and hence  $\mathcal{M}_2^- = [x_2, 2\sigma U]$ . We note that the second derivative of the function in the cosine on the right-hand side of (8.47) satisfies

$$\begin{aligned} \frac{d^2}{dx_0^2} \left( \frac{1}{8} - \frac{2 \left| -\delta x_0^{2/3}/Y + \alpha h \right|^{3/2}}{3^{3/2} \beta^{1/2}} \right) &= \frac{d^2}{dx_0^2} \left( \frac{1}{8} - \frac{2 (\delta x_0^{2/3}/Y - \alpha h)^{3/2}}{3^{3/2} \beta^{1/2}} \right) \\ &= -\frac{2\delta\alpha h}{3^{5/2} x_0^{4/3} Y \beta^{1/2}} \left( \frac{\delta x_0^{2/3}}{Y} - \alpha h \right)^{-1/2}. \end{aligned}$$

Now we write the cosine on the right-hand side of (8.47) in the form  $\cos(2\pi x) = (e(x) + e(-x))/2$ , combine the resulting exponential terms with the term

$$e \left( -\frac{2f_2 d_1 d_2 u x_0}{q} \right),$$

use integration by parts to remove the smooth weight

$$W\left(\frac{x_0}{\sigma U}\right) \exp\left(-\frac{\pi \left|-\delta x_0^{2/3}/Y + \alpha h\right|}{3\beta}\right) \quad (8.50)$$

and employ Lemma 3.2 in [18] with  $k = 2$ . This leads to the conclusion that

$$\int_{x_2}^z Z(x_0) dx_0 \ll \frac{z^{2/3} Y^{1/2} \beta^{1/4}}{\delta^{1/2} \alpha^{1/2} |h|^{1/2}} \left(\frac{\delta z^{2/3}}{Y} - \alpha h\right)^{1/4}.$$

Combining this with (8.49) we get

$$\int_{\mathcal{M}_2^-} \frac{Z(x_0)}{\left|-\delta x_0^{2/3}/Y + \alpha h\right|^{1/4}} dx_0 \ll \frac{\beta^{1/4}}{\alpha^{1/2} |h|^{1/2}} \left( \frac{(\sigma U)^{2/3} Y^{1/2}}{\delta^{1/2}} + \frac{(\sigma U)^{1/3} \delta^{1/2}}{Y^{1/2}} \int_{x_2}^{2\sigma U} \frac{dz}{\delta z^{2/3}/Y - \alpha h} \right).$$

The integral on the right-hand side of this equals that on the left-hand side of (8.48). Hence we obtain

$$\int_{\mathcal{M}_2^-} \frac{Z(x_0)}{\left|-\delta x_0^{2/3}/Y + \alpha h\right|^{1/4}} dx_0 \ll \frac{(\sigma U)^{2/3} Y^{1/2} \beta^{1/4}}{\delta^{1/2} \alpha^{1/2} |h|^{1/2}} \cdot \log U. \quad (8.51)$$

To proceed we will make use of the fact that we can assume that

$$\frac{\delta(\sigma U)^{2/3}}{Y} \ll \beta(\log B)^{1+\varepsilon_0} \quad (8.52)$$

for any  $\varepsilon_0 > 0$ . This is seen as follows. The contribution of the terms with

$$|u| \geq \frac{q}{d_1 d_2 X} \cdot (\log B)^{(1+\varepsilon_0)/2}$$

to  $\mathcal{S}_1^{(1)}$  is negligible, since for such  $u$  we have

$$\Gamma\left(\frac{2^{f_2} d_1 d_2 u X}{q}\right) \ll \exp(-c_1(\log B)^{1+\varepsilon_0}) \quad (8.53)$$

for some constant  $c_1$ , which is  $\ll B^{-c_2}$  for any fixed  $c_2 > 0$ . Hence, we can assume that

$$|u| < \frac{q}{d_1 d_2 X} \cdot (\log B)^{(1+\varepsilon_0)/2}.$$

Then using (5.20), (6.24), (8.3), (8.17), (8.25) and (8.36), we deduce that

$$\beta(\log B)^{1+\varepsilon_0} \gg \frac{aq^2}{bu^2 d_1^2 d_2^2 Y^3} \cdot (\log B)^{1+\varepsilon_0} \gg \frac{aX^2}{bY^3} \gg \frac{a\sigma^2 U}{b\tau^3 T} = \frac{U}{T} \gg \frac{\tau U^{2/3}}{Y} = \frac{\delta(\sigma U)^{2/3}}{Y}.$$

Combining (8.51) with (8.46) and (8.48), we get

$$\int_{\mathcal{M}_2^-} \mathcal{J}_h(x_0) dx_0 \ll \left( \frac{\sigma U}{\alpha |h|} + \frac{(\sigma U)^{2/3} Y^{1/2}}{\delta^{1/2} \alpha^{1/2} |h|^{1/2}} \right) \log U, \quad (8.54)$$

with the convention that the second term can be removed unless  $\alpha|h| \leq \beta(\log B)^{1+\varepsilon}$ . This convention is justified by the observation that if  $\alpha|h| > \beta(\log B)^{1+\varepsilon}$  then the integral in (8.51) becomes negligible. To see this, we bound the weight function in (8.50) similarly as in (8.53) upon noting that the term  $\alpha h$  dominates the term  $\delta x_0^{2/3}/Y$  if  $\alpha|h| > \beta(\log B)^{1+\varepsilon}$  and (8.52) holds with  $\varepsilon_0 < \varepsilon$ .

The same estimate as in (8.54) can be established for the corresponding integral over  $\mathcal{M}_2^+$  in a similar way. In fact  $\mathcal{M}_2^+$  is handled more easily than  $\mathcal{M}_2^-$  since the role of (3.12) is replaced by the simpler (3.6). Now, using  $\mathcal{M}_2 = \mathcal{M}_2^+ \cup \mathcal{M}_2^-$ , we have

$$\int_{\mathcal{M}_2} \mathcal{I}_h(x_0) dx_0 \ll \left( \frac{\sigma U}{\alpha|h|} + \frac{(\sigma U)^{2/3} Y^{1/2}}{\delta^{1/2} \alpha^{1/2} |h|^{1/2}} \right) \log U,$$

where the second term can be removed unless  $\alpha|h| \leq \beta(\log B)^{1+\varepsilon}$ . Hence

$$\begin{aligned} \sum_{1 \leq |h| \leq B^\kappa} \left| \int_{\mathcal{M}_2} \mathcal{I}_h(x_0) dx_0 \right| (h, b_1)^{1/2} \\ \ll \left( \frac{\sigma U}{\alpha} \sum_{1 \leq |h| \leq B^\kappa} \frac{(h, b_1)^{1/2}}{|h|} + \frac{(\sigma U)^{2/3} Y^{1/2}}{\delta^{1/2} \alpha^{1/2}} \sum_{1 \leq |h| \leq \beta(\log B)^{1+\varepsilon}/\alpha} \frac{(h, b_1)^{1/2}}{|h|^{1/2}} \right) \log U \\ \ll \left( \frac{\sigma U}{\alpha} + \frac{(\sigma U)^{2/3} \beta^{1/2} Y^{1/2}}{\delta^{1/2} \alpha} \right) (1 + \varepsilon)^{\omega(b_1)} (\log B)^{1+\varepsilon} \log U. \end{aligned}$$

Combining this with (8.40) and (8.45) we get

$$\mathcal{L} \ll \left( \frac{U}{\alpha T} + \frac{U^{2/3} \beta^{1/2} Y^{1/2}}{\alpha \delta^{1/2} \sigma^{1/3} T} \right) (1 + \varepsilon)^{\omega(b_1)} (\log B)^{1+\varepsilon} \log U, \quad (8.55)$$

for the expression in (8.39).

We compare (8.55) with (6.29) for  $\log P \ll \log B$  and  $\gamma = 0$  (and hence  $Y_0 = Y$ ). The first term on the right-hand side of (8.55) equals the term

$$\frac{\log B}{\alpha} \cdot (1 + \varepsilon)^{\omega(b_1)} \quad (8.56)$$

on the right-hand side of (6.29) times a factor of size

$$\ll \frac{U}{T} \cdot (\log B)^\varepsilon \log U. \quad (8.57)$$

The second term on the right-hand side of (8.55) equals the term

$$\frac{\beta^{1/2}}{\alpha} \cdot (1 + \varepsilon)^{\omega(b_1)} \quad (8.58)$$

on the right-hand side of (6.29) times a factor of size

$$\ll \frac{U^{2/3} Y^{1/2}}{\delta^{1/2} \sigma^{1/3} T} \cdot (\log B)^{1+\varepsilon} \log U = \frac{U^{2/3} Y^{1/2}}{\tau^{1/2} T} \cdot (\log B)^{1+\varepsilon} \log U. \quad (8.59)$$

The term (8.56) in (6.29) gave rise to the first term on the right-hand side of (7.4), and the term (8.58) in (6.29) gave rise to the second term on the right-hand side of (7.4). Hence  $\mathcal{S}_1^{(1)}(U)$  is majorised by the sum of the same terms, multiplied by the factors in (8.57) and (8.59), respectively. In this way we obtain

$$\mathcal{S}_1^{(1)}(U) \ll s^{1/2} s_1^{1/2} q^{1/2} (2 + \varepsilon)^{\omega(b) + \omega(q)} 4^{\omega(s)} (\log B)^{3+\varepsilon} \left( \frac{b^{1/2} Y U}{X T} + \frac{a^{1/2} U^{2/3}}{\tau^{1/2} T} \right) \log U. \quad (8.60)$$

**8.4. Conclusion.** — Combining (8.32), (8.33), (8.34), (8.37), (8.38) and (8.60), we get

$$\begin{aligned} \mathcal{S}(U) &\ll \frac{\varphi(q)}{q^2} \cdot XY \cdot \frac{U}{T} + \frac{XY \operatorname{rad}(q)^{1/2} (2+\varepsilon)^{\omega(q)} U^{1/3} \log B}{\tau q T} \\ &\quad + \frac{(2+\varepsilon)^{\omega(b)} (3+\varepsilon)^{\omega(q)} (\log B)^{2+\varepsilon}}{b^{1/4} q^{1/2} T} \left( \frac{a^{1/2} XY U^{1/3}}{\tau} + \frac{a^{1/4} b^{1/4} X^{1/2} Y U^{5/6}}{\tau^{1/4}} \right) \\ &\quad + s^{1/2} s_1^{1/2} q^{1/2} (2+\varepsilon)^{\omega(b)+\omega(q)} 4^{\omega(s)} (\log B)^{3+\varepsilon} \left( \frac{b^{1/2} Y U}{X T} + \frac{a^{1/2} U^{2/3}}{\tau^{1/2} T} \right) \log U. \end{aligned}$$

In view of (8.3), (8.17), (8.24) and (8.25), we deduce that

$$\begin{aligned} \mathcal{S}(U) &\ll \frac{\varphi(q)}{q} \cdot \frac{B^{5/6} T U^{2/3}}{a^{1/2} b^{1/3} q^{1/6}} + f_\varepsilon(a, b; q) B^{1/2} T^{1/2} U^{1/2} (\log B)^{2+\varepsilon} \\ &\quad + \frac{a^{1/2} b^{1/6} s^{1/2} s_1^{1/2} q^{1/3} (2+\varepsilon)^{\omega(q)+\omega(b)} 4^{\omega(s)} (\log B)^{3+\varepsilon} U^{2/3} \log U}{B^{1/6} T}, \end{aligned} \quad (8.61)$$

where  $f_\varepsilon(a, b; q)$  is given by (8.18).

Now we set

$$Z := \frac{a^{1/2} (b q s s_1)^{1/4}}{B^{1/2}}.$$

We wish to balance the first and the last term on the right-hand side of (8.61). To this end, we choose

$$T := \max \{U^{-1}, \min\{U^{1-\varepsilon}, Z\}\}.$$

This is clearly in accordance with (8.24) and leads to the conclusion in (8.61) that

$$\begin{aligned} \mathcal{S}(U) &\ll \frac{\varphi(q)}{q} \cdot \frac{B^{5/6} U^{-1/3}}{a^{1/2} b^{1/3} q^{1/6}} + f_\varepsilon(a, b; q) B^{1/2} (\log B)^{2+\varepsilon} \left( 1 + \frac{a^{1/4} (b q s s_1)^{1/8} U^{1/2}}{B^{1/4}} \right) \\ &\quad + \frac{a^{1/2} b^{1/6} (s s_1)^{1/2} q^{1/3} (2+\varepsilon)^{\omega(b)} (2+\varepsilon)^{\omega(q)} 4^{\omega(s)} (\log B)^{3+\varepsilon} \log U}{B^{1/6} U^{1/3-\varepsilon}} \\ &\quad + \frac{(s s_1)^{1/4} q^{1/12} (2+\varepsilon)^{\omega(b)+\omega(q)} 4^{\omega(s)} B^{1/3} (\log B)^{4+\varepsilon} U^{2/3}}{b^{1/12}}. \end{aligned}$$

Recalling §8.2, and in particular (8.20), (8.31) and the need to sum over the  $O(\log B)$  dyadic intervals for  $U$  satisfying (8.21), we deduce that

$$\begin{aligned} \mathcal{S}(\mathcal{R}_2; a, b; q) &\ll \frac{\varphi(q)}{q} \cdot \frac{B^{5/6} L^{-1/6}}{a^{1/2} b^{1/3} q^{1/6}} + f_\varepsilon(a, b; q) B^{1/2} (\log B)^{3+\varepsilon} \\ &\quad + f_\varepsilon(a, b; q) a^{1/4} (b q s s_1)^{1/8} B^{1/4} (\log B)^{2+\varepsilon} U^{1/2} \\ &\quad + a^{1/2} b^{1/6} (s s_1)^{1/2} q^{1/3} (2+\varepsilon)^{\omega(b)} (2+\varepsilon)^{\omega(q)} 4^{\omega(s)} B^{-1/6} (\log B)^{3+\varepsilon} \\ &\quad + \frac{(s s_1)^{1/4} q^{1/12} (2+\varepsilon)^{\omega(b)+\omega(q)} 4^{\omega(s)} B^{1/3} (\log B)^{4+\varepsilon} U^{2/3}}{b^{1/12}}. \end{aligned} \quad (8.62)$$

On combining (8.12), (8.19) and (8.62), and taking (8.10) into account, we get a final estimate for  $\mathcal{S}(\mathcal{R}; a, b; q)$ . In view of (8.7) and (8.22) we may now record our final asymptotic formula for  $M(B, \mathbf{X}, \mathbf{Y}; a, b; q)$  in the following result.

**Theorem 8.1.** — *Let  $\varepsilon > 0$ , let  $B \geq e^e$ ,  $\mathbf{X}, \mathbf{Y} \geq 1$  and assume that  $(ab, q) = 1$  are such that (8.1) holds. Let  $\operatorname{rad}(q), s, s_1$  be given by (1.5) and recall the definition (8.3) of  $\sigma, \tau$ . Suppose that*

$$\frac{\mathbf{X}}{\sigma} \geq 1, \quad \frac{\mathbf{Y}}{\tau} \geq 1,$$

and let  $\mathcal{R}$  be given by (8.8). Then we have

$$M(B, \mathbf{X}, \mathbf{Y}; a, b; q) = \frac{\varphi(q)}{q} \cdot \frac{B^{5/6} \text{meas}(\mathcal{R})}{a^{1/2} b^{1/3} q^{1/6}} \left( 1 + O\left( \frac{1}{(\log \log B)^{1/6}} \right) \right) + O\left( \sum_{i=1}^4 F_i \right),$$

where if  $U = \min \left\{ \mathbf{X}/\sigma, (\mathbf{Y}/\tau)^{3/2} \right\}$  and  $f_\varepsilon(a, b; q)$  is given by (8.18), then

$$\begin{aligned} F_1 &:= f_\varepsilon(a, b; q) B^{1/2} (\log B)^{3+\varepsilon}, \\ F_2 &:= f_\varepsilon(a, b; q) a^{1/4} (b q s s_1)^{1/8} B^{1/4} (\log B)^{2+\varepsilon} U^{1/2}, \\ F_3 &:= a^{1/2} b^{1/6} (s s_1)^{1/2} q^{1/3} (2 + \varepsilon)^{\omega(b)} (2 + \varepsilon)^{\omega(q)} 4^{\omega(s)} B^{-1/6} (\log B)^{3+\varepsilon}, \\ F_4 &:= b^{-1/12} (s s_1)^{1/4} q^{1/12} (2 + \varepsilon)^{\omega(b) + \omega(q)} 4^{\omega(s)} B^{1/3} (\log B)^{4+\varepsilon} U^{2/3}. \end{aligned}$$

## 9. Proof of Theorem 1.1

The remaining sections of this paper are concerned with the proof of Theorem 1.1. Let  $U = X \setminus \{x_0 = x_1 = 0\}$  be the distinguished open subset of our degree 2 del Pezzo surface  $X$  in (1.2). Then we see that the counting function is

$$N_U(B) = \frac{1}{2} \# \left\{ \mathbf{x} \in \mathbb{Z}^4 : \begin{array}{l} x_0^2 = x_1 x_2^3 + x_1^3 x_3, \ x_1 \neq 0, \\ (x_1, x_2, x_3) = 1, \\ |x_i| \leq B \text{ for } 1 \leq i \leq 3, \ |x_0| \leq B^2 \end{array} \right\}, \quad (9.1)$$

on taking into account that  $-\mathbf{x}$  and  $\mathbf{x}$  represent the same point in  $\mathbb{P}(2, 1, 1, 1)$ .

**9.1. Passage to the universal torsor.** — We begin by establishing an explicit bijection between the rational points which have to be counted on  $U$  and the integral points on the universal torsor above  $\tilde{X}$  which are subject to a number of coprimality conditions. We begin by simplifying the expression for  $N_U(B)$  in (9.1). The contribution from solutions for which  $x_0 = 0$  is bounded by the number of points of height at most  $B$  on the plane cubic  $x_2^3 + x_1^2 x_3 = 0$ . But the solutions of this equation are parametrised by  $(x_1, x_2, x_3) = (s^3, -s^2 t, t^3)$  and so such points contribute  $O(B^{2/3})$  to  $N_U(B)$ . Turning to solutions with  $x_0 x_1 \neq 0$  we note that  $x_1(x_2^3 + x_1^2 x_3) > 0$  in any solution to be counted. Thus we may assume without loss of generality that  $x_0, x_1 > 0$  on multiplying the final count by 4. We record the outcome of these manipulations in the following result.

**Lemma 9.1.** — *We have*

$$N_U(B) = 2 \# \left\{ \mathbf{x} \in \mathbb{Z}^4 : \begin{array}{l} x_0^2 = x_1 x_2^3 + x_1^3 x_3, \ x_0, x_1 > 0, \\ (x_1, x_2, x_3) = 1, \\ |x_i| \leq B \text{ for } 1 \leq i \leq 3, \ |x_0| \leq B^2 \end{array} \right\} + O(B^{2/3}).$$

Our goal is to pass from Lemma 9.1 to a counting function that involves counting suitably restricted integer points on the associated universal torsor. In the present setting the universal torsor is an open subset of the hypersurface (1.3) in  $\mathbb{A}^{11}$ . Let  $\boldsymbol{\eta} = (\eta_1, \dots, \eta_8)$  and  $\boldsymbol{\alpha} = (\alpha_1, \alpha_2, \alpha_3)$ . We let  $\mathcal{N}(B)$  denote the number of integral solutions  $(\boldsymbol{\eta}, \boldsymbol{\alpha}) \in \mathbb{Z}^{11}$  of the equation (1.3) satisfying

$$\eta_1, \dots, \eta_8, \alpha_2 \geq 1, \quad (9.2)$$

together with the height conditions

$$\begin{cases} |\eta_1^2 \eta_2^4 \eta_3^6 \eta_4^5 \eta_5^4 \eta_6^3 \eta_7^3 \eta_8^2| \leq B, & |\eta_1^2 \eta_2^3 \eta_3^4 \eta_4^3 \eta_5^2 \eta_6 \eta_7^2 \alpha_1| \leq B, \\ |\alpha_3| \leq B, & |\eta_1^3 \eta_2^6 \eta_3^9 \eta_4^7 \eta_5^5 \eta_6^3 \eta_7^5 \eta_8 \alpha_2| \leq B^2, \end{cases} \quad (9.3)$$

and the coprimality conditions

$$(\alpha_1, \eta_2 \cdots \eta_8) = (\alpha_2, \eta_1 \cdots \eta_6 \eta_8) = 1, \quad (9.4)$$



$$(\alpha_3, \eta_1 \cdots \eta_7) = 1, \quad (9.5)$$

and

$$\begin{cases} (\eta_8, \eta_1 \cdots \eta_5 \eta_7) = (\eta_7, \eta_1 \eta_2 \eta_4 \eta_5 \eta_6) = (\eta_6, \eta_1 \cdots \eta_4) = 1, \\ (\eta_5, \eta_1 \eta_2 \eta_3) = (\eta_4, \eta_1 \eta_2) = (\eta_3, \eta_1) = 1. \end{cases} \quad (9.6)$$

We will establish the following result.

**Lemma 9.2.** — *We have  $N_U(B) = 2\mathcal{N}(B) + O(B^{2/3})$ .*

*Proof.* — Let  $a, b \in \mathbb{N}$ . During the course of our argument we will make repeated use of the following two facts. Firstly  $a \mid b^2$  if and only if

$$a = ha'^2, \quad b = ha'b', \quad (a', b') = 1,$$

and secondly,  $a \mid b^3$  if and only if

$$a = hk^2a'^3, \quad b = hka'b', \quad (a', h) = (a'k, b') = 1.$$

The second fact is readily deduced from the first after drawing out the greatest common divisor of  $a$  and  $b$ . To establish the first fact we set  $h' = (a, b)$  and  $a = h'a'$ ,  $b = h'b'$  for coprime  $(a', b') = 1$ . But then  $a' \mid h'$  and the claim follows on writing  $h' = a'h$  for  $h \in \mathbb{N}$ .

In what follows we set  $\alpha_3 = x_3$ . For any  $\mathbf{x}$  counted by  $N_U(B)$  we have  $x_1 \mid x_0^2$ . Hence we may write

$$x_0 = hx'_0\eta_8, \quad x_1 = h\eta_8^2, \quad (x'_0, \eta_8) = (h\eta_8^2, x_2, \alpha_3) = 1,$$

for  $h, x'_0, \eta_8 \in \mathbb{N}$ . Once the substitution is made we obtain the new equation  $hx'_0 = x_2^3 + h^2\eta_8^4\alpha_3$ . It now follows that  $h \mid x_2^3$  and so we can write

$$h = jk^2\eta_6^3, \quad x_2 = jk\eta_6\alpha_1,$$

with  $j, k, \eta_6 \in \mathbb{N}$  such that  $\alpha_1 \neq 0$  and  $(\eta_6, j) = (k\eta_6, \alpha_1) = 1$ , in conjunction with the previous coprimality conditions. Substituting back into the equation yields  $x_0^2 = j^2k\alpha_1^3 + jk^2\eta_6^3\eta_8^4\alpha_3$ . Writing  $\ell = (j, k, x'_0)$  and noting that  $\ell^3 \mid x_0^2$ , so that  $\ell \mid (x'_0/\ell)^2$ , we may therefore make the change of variables

$$j = m\eta_3^2j', \quad k = m\eta_3^2k', \quad x'_0 = m^2\eta_3^3x''_0,$$

for  $m, \eta_3, j', k', x''_0 \in \mathbb{N}$  satisfying  $(j', k', m\eta_3x''_0) = (\eta_3, x''_0) = 1$ , in addition to the previous conditions. This substitution now leads to the new equation  $m^2x''_0 = j'^2k'\alpha_1^3 + j'k'^2\eta_6^3\eta_8^4\alpha_3$ . We are now led to the change of variables

$$j' = \eta_2j'', \quad k' = \eta_4k'', \quad m = \eta_2\eta_4\eta_7,$$

for  $\eta_2, \eta_4, \eta_7, j'', k'' \in \mathbb{N}$  with  $(j''k'', \eta_7) = 1$ , together with all the previous coprimality conditions. This yields the equation  $\eta_7x''_0 = \eta_2j''^2k''\alpha_1^3 + \eta_4j''k''^2\eta_6^3\eta_8^4\alpha_3$ . Now  $j''k'' \mid \eta_7x''_0$  and so  $j''k'' \mid x''_0^2$ . Moreover we have  $(j'', k'') = 1$  since clearly  $(j'', k'', x''_0) = 1$ . We are now led to make our final change of variables

$$j'' = u\eta_1^2, \quad k'' = v\eta_5^2, \quad x''_0 = uv\eta_1\eta_5\alpha_2,$$

for  $u, v, \eta_1, \eta_5, \alpha_2 \in \mathbb{N}$  such that  $(u\eta_1, v\eta_5) = (uv\eta_1\eta_5, \alpha_2) = 1$ . Substituting this back in gives the equation  $uv\eta_7\alpha_2^2 = u\eta_1^2\eta_2\alpha_1^3 + v\eta_4\eta_5^2\eta_6^3\eta_8^4\alpha_3$ . But then  $u \mid \eta_4\eta_5^2\eta_6^3\eta_8^4\alpha_3$  and  $v \mid \eta_1^2\eta_2\alpha_1^3$ . Consideration of the coprimality conditions leads to the conclusion that  $u = v = 1$ , and the equation simply becomes the universal torsor recorded in (1.3).

Tracing through our argument one sees that we have made the transformation

$$\begin{aligned} x_0 &= \eta_1^3 \eta_2^6 \eta_3^9 \eta_4^7 \eta_5^5 \eta_6^3 \eta_7^5 \eta_8 \alpha_2, \\ x_1 &= \eta_1^2 \eta_2^4 \eta_3^6 \eta_4^5 \eta_5^4 \eta_6^3 \eta_7^3 \eta_8^2, \\ x_2 &= \eta_1^2 \eta_2^3 \eta_3^4 \eta_4^3 \eta_5^2 \eta_6 \eta_7^2 \alpha_1, \\ x_3 &= \alpha_3, \end{aligned}$$

with  $(\boldsymbol{\eta}, \boldsymbol{\alpha}) \in \mathbb{Z}^{11}$  satisfying (9.2). This transformation leads directly to the height conditions in (9.3). Furthermore, a straightforward calculation shows that once taken in the light of (1.3) the final coprimality conditions are given by (9.4), (9.5) and (9.6).

Finally it is trivial to check that through the above transformation any point solution  $(\boldsymbol{\eta}, \boldsymbol{\alpha}) \in \mathbb{Z}^{11}$  of (1.3) satisfying (9.2)–(9.6) produces a vector  $\mathbf{x} \in \mathbb{Z}^4$  that will be counted by  $N_U(B)$ . This completes the proof of the lemma.  $\square$

**9.2. Counting integral points on the universal torsor.** — We aim to establish an asymptotic formula for the quantity  $\mathcal{N}(B)$  defined in §9.1. Accordingly we will assume that  $B$  is large in all that follows. To simplify the situation, we define

$$\begin{aligned} X_0 &:= \left( \frac{\eta_1^2 \eta_2^4 \eta_3^6 \eta_4^5 \eta_5^4 \eta_6^3 \eta_7^3 \eta_8^2}{B} \right)^{1/3}, & X_1 &:= \left( \frac{B \eta_4 \eta_5^2 \eta_6^3 \eta_8^4}{\eta_1^2 \eta_2} \right)^{1/3}, \\ X_2 &:= (B \eta_1 \eta_2^2 \eta_3^3 \eta_4^4 \eta_5^5 \eta_6^6 \eta_8^7)^{1/3}. \end{aligned} \tag{9.7}$$

Then one sees that the height conditions (9.2) and (9.3) can be reformulated as

$$0 < X_0 \leq 1, \quad \left| \alpha_1 \cdot \frac{X_0^2}{X_1} \right| \leq 1, \quad 0 < \alpha_2 \cdot \frac{X_0^5}{X_2} \leq 1, \quad |\alpha_3| \leq B,$$

with  $\eta_1, \dots, \eta_8 \geq 1$ .

Once taken together with (1.3) it is easy to see that the coprimality conditions (9.4)–(9.6) are equivalent to the same conditions, but with (9.5) replaced by

$$(\alpha_3, \eta_3 \cdots \eta_6) = 1.$$

Let us write  $S(B, \mathbf{X}, \mathbf{Y}; a, b; q)$  for the set underpinning the cardinality introduced in (1.4). Then it follows from Möbius inversion that

$$\mathcal{N}(B) = \sum_{\substack{\boldsymbol{\eta} \in \mathbb{N}^8 \\ (9.6) \text{ holds} \\ X_0 \leq 1}} \sum_{k_3 \mid \eta_3 \cdots \eta_6} \mu(k_3) N_{k_3},$$

where

$$N_{k_3} := \# \left\{ (x, y) \in S \left( \frac{B}{k_3}, \frac{X_2}{X_0^5}, \frac{X_1}{X_0^2}; \eta_7, \eta_1^2 \eta_2; k_3 \eta_4 \eta_5^2 \eta_6^3 \eta_8^4 \right) : \begin{array}{l} (x, \eta_1 \eta_2 \eta_3) = 1, \\ (y, \eta_2 \eta_3 \eta_7) = 1 \end{array} \right\}.$$

Note that  $x = \alpha_2$  and  $y = \alpha_1$  in this correspondence. The coprimality condition  $(k_3, xy) = 1$  is automatic, since  $k_3 \mid \eta_3 \cdots \eta_6$  and  $(xy, \eta_3 \cdots \eta_6) = 1$ . Furthermore, we clearly have  $(\eta_1^2 \eta_2 \eta_7, k_3 \eta_4 \eta_5^2 \eta_6^3 \eta_8^4) = 1$ . In particular  $N_{k_3} = 0$  unless  $(k_3, \eta_1 \eta_2 \eta_7) = 1$ . Since  $k_3 \mid \eta_3 \cdots \eta_6$  and  $(\eta_1, \eta_3 \cdots \eta_6) = 1$  we see that  $k_3$  is automatically coprime to  $\eta_1$ , whence

$$\mathcal{N}(B) = \sum_{\substack{\boldsymbol{\eta} \in \mathbb{N}^8 \\ (9.6) \text{ holds} \\ X_0 \leq 1}} \sum_{\substack{k_3 \mid \eta_3 \cdots \eta_6 \\ (k_3, \eta_2 \eta_7) = 1}} \mu(k_3) N_{k_3}.$$

A further application of Möbius inversion therefore leads to the expression

$$\mathcal{N}(B) = \sum_{\substack{\boldsymbol{\eta} \in \mathbb{N}^8 \\ (9.6) \text{ holds} \\ X_0 \leq 1}} \sum_{\substack{k_3 | \eta_3 \cdots \eta_6 \\ (k_3, \eta_2 \eta_7) = 1}} \mu(k_3) \sum_{\substack{k_1 | \eta_2 \eta_3 \eta_7 \\ k_2 | \eta_1 \eta_2 \eta_3 \\ (k_1 k_2, k_3 \eta_4 \eta_5 \eta_6 \eta_8) = 1}} \mu(k_1) \mu(k_2) N_{k_1, k_2, k_3}, \quad (9.8)$$

where

$$N_{k_1, k_2, k_3} := M \left( \frac{B}{k_3}, \frac{X_2}{k_2 X_0^5}, \frac{X_1}{k_1 X_0^2}; k_2^2 \eta_7, k_1^3 \eta_1^2 \eta_2; k_3 \eta_4 \eta_5^2 \eta_6^3 \eta_8^4 \right),$$

in the notation of (1.4).

Thus we are led to apply Theorem 8.1 with  $B$  replaced by  $B/k_3$ ,

$$a = k_2^2 \eta_7 \in \mathbb{N}, \quad b = k_1^3 \eta_1^2 \eta_2 \in \mathbb{N}, \quad q = k_3 \eta_4 \eta_5^2 \eta_6^3 \eta_8^4 \in \mathbb{N},$$

and

$$\mathbf{X} = \frac{X_2}{k_2 X_0^5} \geq 1, \quad \mathbf{Y} = \frac{X_1}{k_1 X_0^2} \geq 1 \quad (9.9)$$

One readily checks that  $(ab, q) = 1$  and the inequalities in (9.9) are satisfied by our height conditions. Moreover, consultation with (8.3) and (9.7) reveals that

$$\begin{aligned} \frac{\mathbf{X}}{\sigma} &= \frac{X_2}{k_2 X_0^5} \left( \frac{k_2^2 \eta_7}{\eta_4 \eta_5^2 \eta_6^3 \eta_8^4 B} \right)^{1/2} = X_0^{-9/2} \\ \frac{\mathbf{Y}}{\tau} &= \frac{X_1}{k_1 X_0^2} \left( \frac{k_1^3 \eta_1^2 \eta_2}{\eta_4 \eta_5^2 \eta_6^3 \eta_8^4 B} \right)^{1/3} = X_0^{-2}. \end{aligned}$$

It follows that

$$\frac{\mathbf{X}}{\sigma} \geq \left( \frac{\mathbf{Y}}{\tau} \right)^{3/2} = X_0^{-3} \geq 1,$$

since  $X_0 \leq 1$ . Hence  $\mathbf{U} = X_0^{-3}$ . Thus all the hypotheses of Theorem 8.1 are met and we are therefore free to apply this result. To do so we note from (1.5) that in the present setting we have

$$s \mid \frac{k_3 \eta_4}{(k_3, \eta_6)}, \quad s_1 \mid \frac{k_3 \eta_4 \eta_6}{(k_3, \eta_6)},$$

whence in particular

$$ss_1 \leq \frac{k_3^2 \eta_4^2 \eta_6}{(k_3, \eta_6)^2}.$$

Furthermore (8.18) yields

$$f_\varepsilon(a, b; q) \ll (\eta_2 \cdots \eta_7)^\varepsilon \left( \frac{(2 + \varepsilon)^{\omega(\eta_8)}}{k_2} + \frac{(3 + \varepsilon)^{\omega(\eta_8)}}{\eta_1^{1/2 - \varepsilon}} \right).$$

Let  $\varphi^*(q) = \varphi(q)/q$  be the function introduced in §3.3 and define

$$\gamma(\boldsymbol{\eta}) := \sum_{\substack{k_3 | \eta_3 \cdots \eta_6 \\ (k_3, \eta_2 \eta_7) = 1}} \frac{\mu(k_3)}{k_3} \sum_{\substack{k_1 | \eta_2 \eta_3 \eta_7 \\ k_2 | \eta_1 \eta_2 \eta_3 \\ (k_1 k_2, k_3 \eta_4 \eta_5 \eta_6 \eta_8) = 1}} \frac{\mu(k_1)}{k_1} \cdot \frac{\mu(k_2)}{k_2} \cdot \varphi^*(k_3 \eta_4 \eta_5^2 \eta_6^3 \eta_8^4) \quad (9.10)$$

if (9.6) holds and  $\gamma(\boldsymbol{\eta}) = 0$  otherwise. For  $1 \leq i \leq 4$  let  $\mathcal{F}_i(B)$  denote the overall contribution from the term  $F_i$  in Theorem 8.1 once inserted into (9.8). Then we have

$$\mathcal{N}(B) = \mathcal{M}(B) \left( 1 + O \left( \frac{1}{(\log \log B)^{1/6}} \right) \right) + O \left( \sum_{i=1}^4 \mathcal{F}_i(B) \right), \quad (9.11)$$

where

$$\mathcal{M}(B) := \sum_{\substack{\boldsymbol{\eta} \in \mathbb{N}^8 \\ X_0 \leq 1}} \frac{\text{meas}(\mathcal{R}) B^{5/6} \gamma(\boldsymbol{\eta})}{\eta_1^{2/3} \eta_2^{1/3} \eta_4^{1/6} \eta_5^{1/3} \eta_6^{1/2} \eta_7^{1/2} \eta_8^{2/3}} \quad (9.12)$$

and  $\mathcal{R}$  is given by (8.8).

Let us now turn to the estimation of

$$\mathcal{F}_i(B) \ll \sum_{\substack{\boldsymbol{\eta} \in \mathbb{N}^8 \\ X_0 \leq 1}} \sum_{\substack{k_1 | \eta_2 \eta_3 \eta_7 \\ k_2 | \eta_1 \eta_2 \eta_3 \\ k_3 | \eta_3 \cdots \eta_6}} |\mu(k_1) \mu(k_2) \mu(k_3)| F_i,$$

for  $1 \leq i \leq 4$ . Note that  $F_i$  depends on  $k_1, k_2, k_3$ . During the course of these arguments we will make use of the estimates from §3.3.

Beginning with the case  $i = 1$ , we clearly have

$$\begin{aligned} \mathcal{F}_1(B) &\ll B^{1/2} (\log B)^{3+\varepsilon} \sum_{\substack{\boldsymbol{\eta} \in \mathbb{N}^8 \\ X_0 \leq 1}} \sum_{k_2 | \eta_1 \eta_2 \eta_3} |\mu(k_2)| (\eta_2 \cdots \eta_7)^\varepsilon \left( \frac{(2+\varepsilon)^{\omega(\eta_8)}}{k_2} + \frac{(3+\varepsilon)^{\omega(\eta_8)}}{\eta_1^{1/2-\varepsilon}} \right) \\ &\ll B^{1/2} (\log B)^{3+\varepsilon} \sum_{\substack{\boldsymbol{\eta} \in \mathbb{N}^8 \\ X_0 \leq 1}} (\eta_2 \cdots \eta_7)^\varepsilon \left( (2+\varepsilon)^{\omega(\eta_8)} + \frac{(3+\varepsilon)^{\omega(\eta_8)}}{\eta_1^{1/2-\varepsilon}} \right). \end{aligned}$$

On summing first over  $\eta_8$  and then over  $\eta_1$  we therefore obtain  $\mathcal{F}_1(B) \ll B(\log B)^{5+\varepsilon}$ , which is satisfactory.

For the case  $i = 2$  we first note that

$$\begin{aligned} F_2 &\ll (\eta_2 \cdots \eta_7)^\varepsilon \left( \frac{(2+\varepsilon)^{\omega(\eta_8)}}{k_2} + \frac{(3+\varepsilon)^{\omega(\eta_8)}}{\eta_1^{1/2-\varepsilon}} \right) (k_2^2 \eta_7)^{1/4} (k_1^3 \eta_1^2 \eta_2)^{1/8} \\ &\quad \times (k_3 \eta_4 \eta_5^2 \eta_6^3 \eta_8^4)^{1/8} (k_3^2 \eta_4^2 \eta_6)^{1/8} \left( \frac{B}{k_3} \right)^{1/4} (\log B)^{2+\varepsilon} X_0^{-3/2} \\ &= k_1^{3/8} k_2^{1/2} k_3^{1/8} (\eta_2 \cdots \eta_7)^\varepsilon \left( \frac{(2+\varepsilon)^{\omega(\eta_8)}}{k_2} + \frac{(3+\varepsilon)^{\omega(\eta_8)}}{\eta_1^{1/2-\varepsilon}} \right) \\ &\quad \times \frac{B^{3/4} (\log B)^{2+\varepsilon}}{\eta_1^{3/4} \eta_2^{15/8} \eta_3^3 \eta_4^{17/8} \eta_5^{7/4} \eta_6^{5/4} \eta_7^{1/2} \eta_8}. \end{aligned}$$

Recall that  $k_1, k_2, k_3$  do not depend on  $\eta_8$ . Summing over  $\eta_8 \leq B^{1/2} / (\eta_1 \eta_2^2 \eta_3^3 \eta_4^{5/2} \eta_5^2 \eta_6^{3/2} \eta_7^{3/2})$ , we therefore find that

$$\sum_{\eta_8} F_2 \ll B(\log B)^{4+\varepsilon} \cdot \frac{k_1^{3/8} k_2^{1/2} k_3^{1/8} (\eta_2 \cdots \eta_7)^\varepsilon}{\eta_1^{5/4} \eta_2^{23/8} \eta_3^{9/2} \eta_4^{27/8} \eta_5^{11/4} \eta_6^{7/4} \eta_7^2} \left( \frac{1}{k_2} + \frac{1}{\eta_1^{1/2-\varepsilon}} \right).$$

It is now trivial to confirm that  $\mathcal{F}_2(B) \ll B(\log B)^{4+\varepsilon}$ , which is also satisfactory.

For the case  $i = 3$  we begin by noting that

$$\begin{aligned}
F_3 &\ll (\eta_2 \cdots \eta_7)^\varepsilon (k_2^2 \eta_7)^{1/2} (k_1^3 \eta_1^2 \eta_2)^{1/6} \left( \frac{k_3^2 \eta_4^2 \eta_6}{(k_3, \eta_6)^2} \right)^{1/2} \\
&\quad \times (k_3 \eta_4 \eta_5^2 \eta_6^3 \eta_8^4)^{1/3} (2 + \varepsilon)^{\omega(\eta_1) + \omega(\eta_8)} (B/k_3)^{-1/6} (\log B)^{3+\varepsilon} \\
&= \frac{k_1^{1/2} k_2 k_3^{3/2}}{(k_3, \eta_6)} \cdot (\eta_2 \cdots \eta_7)^\varepsilon (2 + \varepsilon)^{\omega(\eta_1) + \omega(\eta_8)} \\
&\quad \times \frac{\eta_1^{1/3} \eta_2^{1/6} \eta_4^{4/3} \eta_5^{2/3} \eta_6^{3/2} \eta_7^{1/2} \eta_8^{4/3} (\log B)^{3+\varepsilon}}{B^{1/6}}.
\end{aligned}$$

Summing first over  $\eta_8$  we conclude that

$$\begin{aligned}
\sum_{\eta_8} F_3 &\ll B(\log B)^{4+\varepsilon} \cdot \frac{k_1^{1/2} k_2 k_3^{3/2}}{(k_3, \eta_6)} \cdot \frac{(\eta_2 \cdots \eta_7)^\varepsilon (2 + \varepsilon)^{\omega(\eta_1)}}{\eta_1^2 \eta_2^{9/2} \eta_3^7 \eta_4^{9/2} \eta_5^4 \eta_6^2 \eta_7^3} \\
&\ll B(\log B)^{4+\varepsilon} \cdot \frac{k_2 (\eta_2 \cdots \eta_7)^\varepsilon (2 + \varepsilon)^{\omega(\eta_1)}}{\eta_1^2 \eta_2^4 \eta_3^5 \eta_4^3 \eta_5^{5/2} \eta_6^{3/2} \eta_7^{5/2}},
\end{aligned}$$

since  $k_1 \leq \eta_2 \eta_3 \eta_7$  and  $k_3^{3/2}/(k_3, \eta_6) \leq (\eta_3 \eta_4 \eta_5)^{3/2} \eta_6^{1/2}$ . It therefore follows from (3.19) that

$$\begin{aligned}
\mathcal{F}_3(B) &\ll B(\log B)^{4+\varepsilon} \sum_{\eta_1, \dots, \eta_7 \leq B} \frac{(\eta_2 \cdots \eta_7)^\varepsilon (2 + \varepsilon)^{\omega(\eta_1)}}{\eta_1^2 \eta_2^4 \eta_3^5 \eta_4^3 \eta_5^{5/2} \eta_6^{3/2} \eta_7^{5/2}} \sum_{k_2 | \eta_1 \eta_2 \eta_3} |\mu(k_2)| k_2 \\
&\ll B(\log B)^{4+\varepsilon} \sum_{\eta_1 \leq B} \frac{(2 + \varepsilon)^{\omega(\eta_1)} \sigma_1(\eta_1)}{\eta_1^2} \\
&\ll B(\log B)^{6+\varepsilon},
\end{aligned}$$

which is satisfactory.

Finally for  $i = 4$  we note that

$$\begin{aligned}
F_4 &\ll (\eta_2 \cdots \eta_7)^\varepsilon (k_1^3 \eta_1^2 \eta_2)^{-1/12} \left( \frac{k_3^2 \eta_4^2 \eta_6}{(k_3, \eta_6)^2} \right)^{1/4} (k_3 \eta_4 \eta_5^2 \eta_6^3 \eta_8^4)^{1/12} \\
&\quad \times (2 + \varepsilon)^{\omega(\eta_1)} (2 + \varepsilon)^{\omega(\eta_8)} \left( \frac{B}{k_3} \right)^{1/3} (\log B)^{4+\varepsilon} X_0^{-2} \\
&\ll \frac{(\eta_2 \cdots \eta_7)^\varepsilon (2 + \varepsilon)^{\omega(\eta_8)} (2 + \varepsilon)^{\omega(\eta_1)} B(\log B)^{4+\varepsilon}}{\eta_1^{3/2} \eta_2^{11/4} \eta_3^{15/4} \eta_4^{5/2} \eta_5^{9/4} \eta_6^{3/2} \eta_7^2 \eta_8},
\end{aligned}$$

on taking  $k_3/(k_3, \eta_6) \leq \eta_3 \eta_4 \eta_5$ . On summing first over  $\eta_8$  one is readily led to the conclusion that  $\mathcal{F}_4(B) \ll B(\log B)^{6+\varepsilon}$ , which is satisfactory.

Recalling (9.11), we may summarise our investigation so far in the following result.

**Lemma 9.3.** — *We have*

$$\mathcal{N}(B) = \mathcal{M}(B) \left( 1 + O \left( \frac{1}{(\log \log B)^{1/6}} \right) \right) + O(B(\log B)^{6+\varepsilon}),$$

as  $B \rightarrow \infty$ , where  $\mathcal{M}(B)$  is given by (9.12) and (9.10).

It remains to estimate  $\mathcal{M}(B)$  as  $B \rightarrow \infty$ . Let us begin by considering the factor  $\text{meas}(\mathcal{R})$ . For  $u \in (0, 1)$  we write

$$v(u) := \int \int_{\{|s^2 + t^3| \leq 1, 0 < s < u^{-9/2}, |t| \leq u^{-2}\}} ds dt.$$

Then we have  $\text{meas}(\mathcal{R}) = v(X_0)$ . Writing  $n = \eta_1^2 \eta_2^4 \eta_3^6 \eta_4^5 \eta_5^4 \eta_6^3 \eta_7^3 \eta_8^2$ , so that  $X_0 = (n/B)^{1/3}$  in (9.7), we deduce that

$$\mathcal{M}(B) = B^{5/6} \sum_{n \leq B} v \left( (n/B)^{1/3} \right) \Delta(n), \quad (9.13)$$

where

$$\Delta(n) := \sum_{\substack{\boldsymbol{\eta} \in \mathbb{N}^8 \\ n = \eta_1^2 \eta_2^4 \eta_3^6 \eta_4^5 \eta_5^4 \eta_6^3 \eta_7^3 \eta_8^2}} \frac{\gamma(\boldsymbol{\eta}) n^{1/6}}{\eta_1 \eta_2 \eta_3 \eta_4 \eta_5 \eta_6 \eta_7 \eta_8}$$

and  $\gamma$  is given by (9.10). Let us bring  $\gamma$  into a simpler shape under the assumption that (9.6) holds. Recall the definition of  $\varphi^*$  from §3.3 and its properties recorded there. Carrying out the inner summation in (9.10), we get

$$\gamma(\boldsymbol{\eta}) = \sum_{\substack{k_3 | \eta_3 \cdots \eta_6 \\ (k_3, \eta_2 \eta_7) = 1}} \frac{\mu(k_3)}{k_3} \cdot \frac{\varphi^*(\eta_2 \eta_3 \eta_7) \varphi^*(\eta_1 \eta_2 \eta_3) \varphi^*(k_3 \eta_4 \eta_5 \eta_6 \eta_8)}{\varphi^*(\eta_2 \eta_3 \eta_7, k_3 \eta_4 \eta_5 \eta_6 \eta_8) \varphi^*(\eta_1 \eta_2 \eta_3, k_3 \eta_4 \eta_5 \eta_6 \eta_8)}.$$

Since  $\varphi^*(ab) \varphi^*(a, b) = \varphi^*(a) \varphi^*(b)$  we see that the summand is

$$\frac{\mu(k_3)}{k_3} \cdot \frac{\varphi^*(\eta_1 \cdots \eta_6 \eta_8) \varphi^*(\eta_2 \cdots \eta_8)}{\varphi^*(k_3 \eta_4 \eta_5 \eta_6 \eta_8)}.$$

Applying (3.15), we obtain

$$\gamma(\boldsymbol{\eta}) = \frac{\varphi^*(\eta_2 \cdots \eta_8) \varphi^*(\eta_1 \cdots \eta_6 \eta_8) \varphi^*(\eta_4 \eta_5 \eta_6 (\eta_3, \eta_2 \eta_7))}{\varphi^*(\eta_3 \cdots \eta_6 \eta_8)} \prod_{\substack{p | \eta_3 \\ p \nmid \eta_2 \eta_4 \cdots \eta_8}} \left( 1 - \frac{2}{p} \right)$$

if (9.6) holds and  $\gamma(\boldsymbol{\eta}) = 0$  otherwise. Using this it is now straightforward to calculate the corresponding Dirichlet series

$$F(s + 1/6) = \sum_{n=1}^{\infty} \frac{\Delta(n)}{n^s} = \sum_{\boldsymbol{\eta} \in \mathbb{N}^8} \frac{\gamma(\boldsymbol{\eta})}{\eta_1^{2s+1} \eta_2^{4s+1} \eta_3^{6s+1} \eta_4^{5s+1} \eta_5^{4s+1} \eta_6^{3s+1} \eta_7^{3s+1} \eta_8^{2s+1}},$$

which is absolutely convergent for  $\Re(s) > 0$  and features non-negative Dirichlet coefficients. By multiplicativity we have an Euler product  $F(s) = \prod_p F_p(s)$ , and one finds that  $F_p(s + 1/6) = 1 + (1 - 1/p) H_p(s)$ , with

$$\begin{aligned} H_p(s) &= \frac{p^{-(2s+1)}(1 - p^{-(3s+2)})}{(1 - p^{-(2s+1)})(1 - p^{-(3s+1)})} + \frac{p^{-(3s+1)}(1 - p^{-(6s+2)})}{(1 - p^{-(3s+1)})(1 - p^{-(6s+1)})} \\ &\quad + \frac{p^{-(3s+1)}(1 - p^{-1})}{(1 - p^{-(3s+1)})(1 - p^{-(4s+1)})} + \frac{p^{-(4s+1)}(1 - p^{-1})}{(1 - p^{-(4s+1)})(1 - p^{-(5s+1)})} \\ &\quad + \frac{p^{-(5s+1)}(1 - p^{-1})}{(1 - p^{-(5s+1)})(1 - p^{-(6s+1)})} + \frac{p^{-(6s+1)}(1 - 2p^{-1} + p^{-(4s+2)})}{(1 - p^{-(6s+1)})(1 - p^{-(4s+1)})} \\ &\quad + \frac{p^{-(4s+1)}(1 - p^{-1})}{(1 - p^{-(4s+1)})(1 - p^{-(2s+1)})} + \frac{p^{-(2s+1)}}{1 - p^{-(2s+1)}}. \end{aligned}$$

Let  $\mathbf{k} = (2, 4, 6, 5, 4, 3, 3, 2)$ . Then there exists  $\delta_1 > 0$  and a function  $G(s)$  which is holomorphic in the region  $\Re(s) \geq -\delta_1$  for which

$$F(s + 1/6) = G(s) \prod_{j=1}^8 \zeta(k_j s + 1).$$

Thus  $F(s)$  has a meromorphic continuation to the half-plane  $\Re(s) \geq 1/6 - \delta_1$ , with a pole of order 8 at  $s = 1/6$ . Moreover it will be useful to note that

$$G(0) = \prod_p \left(1 - \frac{1}{p}\right)^8 \left(1 + \frac{8}{p} + \frac{1}{p^2}\right). \quad (9.14)$$

To estimate  $\mathcal{M}(B)$  we now have everything in place to apply a standard Tauberian theorem, such as that recorded in work of Chambert-Loir and Tschinkel [8, Appendice A], for example. On noting that  $\lim_{s \rightarrow 1/6} (s - 1/6)^8 F(s) = G(0)/(k_1 \dots k_8)$ , we therefore conclude that

$$\sum_{n \leq t} \Delta(n) = \frac{6G(0)t^{1/6}P(\log t)}{7! \cdot 17280} + O(t^{1/6-\delta_2}),$$

for  $t \geq 1$ , some  $\delta_2 > 0$  and a monic polynomial  $P$  of degree 7.

Write  $c = 6G(0)/(7! \cdot 17280)$  and  $E = B^{1/6}(\log B)^6$  for short. Now we may deduce from an application of partial summation and integration by parts that

$$\begin{aligned} \sum_{n \leq B} v\left((n/B)^{1/3}\right) \Delta(n) &= c \int_1^B v\left((t/B)^{1/3}\right) \frac{d}{dt} \left(t^{1/6}(\log t)^7\right) dt + O(E) \\ &= \frac{1}{2} \cdot c B^{1/6} (\log B)^7 \int_0^1 \frac{v(u)}{\sqrt{u}} du + O(E). \end{aligned}$$

Thus, on returning to our expression for  $\mathcal{M}(B)$  in (9.13), we deduce that

$$\mathcal{M}(B) = \frac{1}{2} \cdot \frac{\omega_\infty G(0)}{87091200} \cdot B(\log B)^7 + O\left(B(\log B)^6\right), \quad (9.15)$$

where

$$\omega_\infty = 6 \int \int \int_{\{|s^2+t^3| \leq 1, 0 < s < u^{-9/2}, 0 < u < 1, |t| \leq u^{-2}\}} \frac{dudsdt}{\sqrt{u}}. \quad (9.16)$$

**9.3. Final deduction of Theorem 1.1.** — Corraling Lemmas 9.2 and 9.3 with (9.15) we have therefore shown that

$$N_U(B) = \frac{\omega_\infty G(0)}{87091200} \cdot B(\log B)^7 \left(1 + O\left(\frac{1}{(\log \log B)^{1/6}}\right)\right),$$

where  $G(0)$  is given by (9.14) and  $\omega_\infty$  by (9.16). It remains to show that the leading constant achieved in this estimate agrees with Peyre's prediction [21].

Let  $\tilde{X}$  denote a minimal desingularisation of  $X$ . According to Peyre we should have  $c_X = \alpha(\tilde{X})\omega_H(\tilde{X})$ , where  $\alpha(\tilde{X})$  is a constant related to the geometry of  $X$  and  $\omega_H(\tilde{X})$  is related to the densities of rational points on  $X$  over  $\mathbb{R}$  and  $\mathbb{Q}_p$  for all primes  $p$ . We claim that  $\alpha(\tilde{X}) = 1/87091200$  and

$$\omega_H(\tilde{X}) = \omega_\infty \prod_p \left(1 - \frac{1}{p}\right)^8 \left(1 + \frac{8}{p} + \frac{1}{p^2}\right),$$

with  $\omega_\infty$  given by (9.16). This will clearly suffice to complete the proof of Theorem 1.1.

Beginning with the local densities it follows from [21] that

$$\begin{aligned} \omega_H(\tilde{X}) &= \lim_{s \rightarrow 1} \left( (s-1)^{\text{rank Pic}(\tilde{X})} L(s, \text{Pic}(\tilde{X})) \right) \omega_\infty \prod_p \frac{\omega_p}{L_p(1, \text{Pic}(\tilde{X}))} \\ &= \omega_\infty \prod_p \left(1 - \frac{1}{p}\right)^8 \omega_p, \end{aligned}$$

since  $L(s, \text{Pic}(\tilde{X})) = \zeta(s)^8$ , where  $\omega_\infty$  and  $\omega_p$  are the real and  $p$ -adic densities of points on  $X$ , respectively. Applying a calculation of Loughran [19, Lemma 2.3] we obtain

$$\omega_p = 1 + \frac{8}{p} + \frac{1}{p^2},$$

as claimed. To compute  $\omega_\infty$  we parametrise the points by writing  $x_3$  as a function of  $x_0, x_1, x_2$  in  $f(\mathbf{x}) = x_0^2 + x_1x_2^3 + x_1^3x_3$ . Since  $\mathbf{x} = -\mathbf{x}$  in  $\mathbb{P}(2, 1, 1, 1)$ , we may assume  $x_0 \geq 0$ . On observing that  $\frac{\partial f}{\partial x_3}(\mathbf{x}) = x_1^3$ , the Leray form  $\omega_L(\tilde{X})$  is given by  $x_1^{-3}dx_0dx_1dx_2$ . Hence

$$\omega_\infty = 2 \int \int \int_{\{|x_1^{-3}(x_0^2+x_1x_2^3)| \leq 1, 0 \leq x_0, x_1 \leq 1, |x_2| \leq 1\}} x_1^{-3}dx_0dx_1dx_2.$$

But then the change of variables  $x_0 = sx_1^{3/2}$ ,  $x_2 = tx_1^{2/3}$  and  $x_1 = u^3$ , easily yields the value of  $\omega_\infty$  given in (9.16).

Turning to the calculation of  $\alpha(\tilde{X})$  we follow the approach of Derenthal, Joyce and Teitler [13]. Thus [13, Theorem 1.3] shows that

$$\alpha(\tilde{X}) = \frac{\alpha(Y)}{\#W(R_{\tilde{X}})},$$

where  $\alpha(Y)$  is the “ $\alpha$ -constant” associated to the split non-singular del Pezzo surface  $Y$  of degree 2 and  $W(R_{\tilde{X}})$  is the Weyl group of the root system  $R_{\tilde{X}}$  whose simple roots are the  $(-2)$ -curves on  $\tilde{X}$ . For us the root system is  $\mathbf{E}_7$  by [13, Remark 5.6], whence an application of [13, Table 2] and [11, Theorem 4] shows that

$$\alpha(\tilde{X}) = \frac{1}{30} \cdot \frac{1}{2^{10} \cdot 3^4 \cdot 5 \cdot 7} = \frac{1}{87091200},$$

as required.

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